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Anisotropic multiple scattering in diffusive media

E Amic†§, J M Luck†‖ and Th M Nieuwenhuizen‡¶

† Service de Physique Théorique, Centre d'Études de Saclay, 91191 Gif-sur-Yvette cedex, France

‡ Van der Waals-Zeeman Institute, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands

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Abstract. The multiple scattering of scalar waves in diffusive media is investigated by means of the radiative transfer equation. This approach, which does not rely on the diffusion approximation, becomes asymptotically exact in the regime of most interest, where the scattering mean free path ℓ is much larger than the wavelength λ_0. Quantitative predictions are derived in that regime, concerning various observables pertaining to optically thick slabs, such as the mean angle-resolved reflected and transmitted intensities, and the width of the enhanced backscattering cone. Special emphasis is put on the dependence of these quantities on the anisotropy of the cross section of the individual scatterers, and on the internal reflections due to the optical index mismatch at the boundaries of the sample. The large index mismatch regime is studied analytically, for arbitrary anisotropic scattering. The regime of very anisotropic scattering, where the transport mean free path ℓ^* is much larger than the scattering mean free path ℓ, is then investigated in detail. The relevant Schwarzschild–Milne equation is solved exactly in the absence of internal reflections.

1. Introduction

The theory of multiple light scattering has been a classical subject of interest for one century, which attracted the attention of many scientists, including Lord Rayleigh, Schwarzschild and Chandrasekhar. Standard books are available, such as those by Chandrasekhar [1], Ishimaru [2], van de Hulst [3] and Sobolev [4]. The discovery of weak-localization effects, and chiefly the enhanced backscattering cone [5], yielded a revival of theoretical and experimental work in the area of multiple scattering in disordered media. Much progress has been done recently in the analysis of speckle fluctuations [6], in analogy with the conductance fluctuations observed in mesoscopic electronic systems [7].

Laboratory experiments are often performed either on solid TiO_2 (white paint) samples, or on suspensions of polystyrene spheres or of TiO_2 grains in fluids. In most cases, the wavelength λ_0 of light in the diffusive medium, the scattering mean free path ℓ, and the thickness L of the sample obey the inequalities $\lambda_0 \ll \ell \ll L$. Multiple scattering is also of interest in biophysics and medical physics, in order to understand the transport of radiation through human and animal tissues. Besides light, all kinds of classical waves undergo multiple scattering in media with a high enough level of disorder, i.e. of inhomogeneity. Well known examples are acoustic and seismic waves. The propagation of electrons in disordered solids also pertains to this area, since quantum mechanics also basically consists in wave propagation.

§ E-mail address: amic@amoco.saclay.cea.fr

‖ E-mail address: luck@amoco.saclay.cea.fr

¶ E-mail address: nieuwenh@phys.uva.nl

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Multiple scattering of waves in disordered media admits the following three levels of theoretical description.

(i) The *macroscopic* approach consists in an effective diffusion equation, which describes the transport of the diffuse (incoherent) intensity $I(\boldsymbol{r}, t)$ at point $\boldsymbol{r}$ at time t. This approximation turns out to be very accurate in the bulk of a turbid medium, and more generally on length scales much larger than the mean free path ℓ. The diffusion approximation yields several interesting predictions, among which we mention the $1/L$-decay of the total transmission through an optically thick slab of thickness $L \gg \ell$, the decay time of the transient response to an incident light pulse, or the memory of a typical speckle pattern when the frequency of light is varied. This approximation also allows for a quantitative prediction of the diffuse image of a small object in transmission [8].

(ii) The *mesoscopic* approach, used by astrophysicists throughout the classical era of the subject, is referred to as radiative transfer theory [1–4]. This theory relies on the radiative transfer equation, which is a local balance equation, similar to the Boltzmann equation in kinetic theory, for the diffuse intensity $I(\boldsymbol{r}, \boldsymbol{n}, t)$, with $\boldsymbol{n}$ being the direction of propagation. This approach leads in a natural way to distinguish between the scattering mean free path ℓ and the transport mean free path ℓ^*, to be defined below. The diffusion approach (i) is recovered in the limit of length scales much larger than ℓ^*.

(iii) The *microscopic* approach consists in expanding the solution of the wave equation in the disordered medium in the form of a diagrammatic Born series. In the regime $\ell \gg \lambda_0$ the leading diagrams can be identified, in analogy with, for example, the theory of disordered superconductors [9]. For the diffuse intensity they are the ladder diagrams, which are built up by pairing one retarded and one advanced propagator following the same path through the disordered sample, i.e. the same ordered sequence of scattering events. This picture agrees with that behind the radiative transfer equation, which is a classical transport equation for the intensity. The ladder diagrams can be summed and yield an integral equation of the Bethe–Salpeter type for the diffuse intensity, which we refer to as the Schwarzschild–Milne equation. The radiative transfer approach (ii) is thus recovered in the weak-disorder ($\ell \gg \lambda_0$) regime. A further step consists in including some of the subleading diagrams, of relative order $\lambda_0/\ell \ll 1$, which account for interference effects between diffusive paths. Among those contributions, the class of maximally crossed diagrams is of particular interest, since it describes the aforementioned enhanced backscattering phenomenon [10, 11].

To summarize this discussion, each of the descriptions mentioned above represents a qualitative improvement with respect to the previous ones (see e.g. [12]). In order to derive quantitative estimates of observables in the regime $\ell \gg \lambda_0$ of interest, it is sufficient to consider radiative transfer theory. The macroscopic approach (i) is clearly insufficient, since we aim, among other things, at a description of the crossover from free propagation to diffusive transport which takes place in the *skin layers* of thickness of order ℓ, near the boundaries of the disordered medium. This phenomenon is, by its very definition, beyond the scope of the diffusion approach. The radiative transfer approach (ii) leads us to study the Schwarzschild–Milne equation (2.26), (2.28). This equation has only been solved exactly in a limited number of cases. Analytical results have been obtained for isotropic scattering of scalar waves [1, 13–15], and in the case where the scattering cross section depends linearly on the cosine of the scattering angle [1, 16]. The case of anisotropic scattering has essentially been investigated by means of general formalism and by numerical methods [2, 3, 17].

For a finite sample, physical observables such as the reflected or transmitted intensity in a given direction depend on the particular realization of the sample under consideration. Such quantities are indeed the results of intricate interference patterns throughout the sample; they only become self-averaging quantities in the limit of a large enough sample. This definition can be made precise by means of the dimensionless conductance $g \sim \mathcal{N}\ell/L$, related to the number $\mathcal{N} \sim \mathcal{A}/\lambda_0^2$ of open channels, where $\mathcal{A}$ is the transverse area of the sample. The self-averaging regime corresponds to $g \gg 1$. The whole distribution of observables is therefore of interest, as long as g is not very large. We mention a recent experiment [18], where the third cumulant of the total transmission, an effect of relative order $1/g^2$, has been measured and compared to theoretical predictions. This third cumulant is of the same order of magnitude as the universal conductance fluctuations, either in electronic [7] or in optical [6] systems. In fact the full distribution of the total transmission through an optically thick slab has been derived recently [19]. In the following we focus our attention on the mean values of physical quantities.

The vector character of electromagnetic waves also introduces its own intricacy. In general four coupled integral equations have to be solved, which are associated with the Stokes parameters of the diffuse light in the medium. These equations have been solved exactly in the case of Rayleigh scattering, i.e. the regime where the size of the scatterers is much smaller than the wavelength [1, 20]. Among specific features pertaining to diffusive light propagation, let us mention the dependence of the backscattering enhancement factor on the polarization states of the incoming and outgoing radiation [21–23], or the progressive destruction of the backscattering peak induced by a magnetic field, due to the Faraday rotation in a magneto-optically active material [24–27].

Furthermore, in practical situations the optical index n_0 of the scattering medium is often different from the index n_1 of the surrounding medium. This index mismatch, measured by the ratio $m = n_0/n_1$, causes reflections at the interfaces. In the regime of a large index mismatch ($m \ll 1$ or $m \gg 1$), the transmission across the interfaces is very small, so that the light is reinjected many times in the diffusive medium. As a consequence, the skin layers become very thick in this regime. More generally, the diffusion approximation works better and better as the index mismatch gets large. Improvements of the diffusion equation have been proposed [28, 29], which take internal reflections into account. It is in fact possible to derive analytical expressions for the reflected and transmitted intensities and other observables in this regime. This asymptotic analysis has been performed in [15] for isotropic scattering. It will be generalized hereafter to the case of general anisotropic scattering. This approach provides accurate results, even for a moderate index mismatch m.

More generally, one of the main goals of this paper is to quantify the dependence of quantities on the anisotropy of the scattering mechanism. It has long been known from radiative transfer theory that two length scales are involved in the case of general anisotropic scattering: the scattering mean free path ℓ, namely the effective distance between two successive scattering events, and the transport mean free path ℓ^*, namely the distance over which radiation loses memory of its direction. Both mean free paths, to be defined more precisely in section 2, depend on details of the scattering mechanism, such as the shape, size and dielectric constant of the scatterers. In the regime of very anisotropic scattering, we have $\ell^* \gg \ell$. The ratio $\tau^* = \ell^*/\ell \gg 1$ will be referred to as the *anisotropy parameter*. In some experimental situations τ^* can be of order ten or larger [30]. The regime of most interest is then $\lambda_0 \ll \ell \ll \ell^* \ll L$.

The setup of this paper is as follows. In section 2 we present some general formalism on radiative transfer theory and we derive the associated Schwarzschild–Milne equation. We show how solutions of the latter equation yield predictions for quantities of interest, such as the diffuse reflected and transmitted angle-resolved intensities. The determination of the shape of the enhanced backscattering cone is also addressed. Section 3 is devoted to the regime of a large index mismatch, for general anisotropy. In section 4 we derive a complete analytical solution of the radiative transfer problem in the regime of very anisotropic scattering ($\tau^* \gg 1$), in the absence of internal reflections. The paper closes up with a discussion in section 5.

2. Generalities on anisotropic multiple scattering

Throughout the following we restrict the analysis to multiple scattering of scalar waves by scatterers located at uncorrelated random positions, in the regime where the scattering mean free path ℓ is much larger than the wavelength λ_0 of radiation in the medium. We have summarized some useful notations and definitions in table 1.

Table 1. Conventions and notations for kinematic and other useful quantities.

	Outside medium	Inside medium
Optical index	n_1	$n_0 = mn_1$
Wavenumber	$k_1 = n_1\omega/c = 2\pi/\lambda_1$	$k_0 = n_0\omega/c = mk_1 = 2\pi/\lambda_0$
Incidence angle	θ	θ'
Parallel wavevector	$p = k_1 \cos\theta = k_0\sqrt{\mu^2 - 1 + 1/m^2}$	$P = k_0 \cos\theta' = k_0\mu$
Total reflection condition	$m < 1$ and $\sin\theta > m$ (i.e. P imaginary)	$m > 1$ and $\sin\theta' > 1/m$ (i.e. p imaginary)
Transverse wavevector	$\lvert\boldsymbol{q}\rvert = q = k_1 \sin\theta = k_0 \sin\theta' = k_0\sqrt{1-\mu^2}$	
Azimuthal angle	φ	
Reflection and transmission coefficients	partial reflection: $\begin{cases} R = \left(\dfrac{P-p}{P+p}\right)^2 = \left(\dfrac{\mu - \sqrt{\mu^2-1+1/m^2}}{\mu + \sqrt{\mu^2-1+1/m^2}}\right)^2 \\ T = \dfrac{4Pp}{(P+p)^2} = \dfrac{4\mu\sqrt{\mu^2-1+1/m^2}}{(\mu+\sqrt{\mu^2-1+1/m^2})^2} \end{cases}$	total reflection: $\begin{cases} R = 1 \\ T = 0 \end{cases}$

As stated in section 1, we put special emphasis on the dependence of physical quantities on the anisotropy of the scattering mechanism. After averaging over the random orientations of the individual scatterers, the differential scattering cross section of arbitrary anisotropic scatterers can be written in the following form [1, 2]

$$\mathrm{d}\sigma(\boldsymbol{n}, \boldsymbol{n}') = (u/4\pi)^2 p(\Theta)\,\mathrm{d}\Omega'. \qquad (2.1)$$

In this formula u is the scattering length, $\boldsymbol{n}$ and $\boldsymbol{n}'$ are unit vectors in the incident and outgoing directions, Θ is the angle between these directions, so that $\cos\Theta = \boldsymbol{n}\cdot\boldsymbol{n}'$, and $\mathrm{d}\Omega'$ is an element of solid angle around the direction $\boldsymbol{n}'$. We assume that there is neither absorption nor inelastic scattering. In other words the albedo is unity (the situation of non-conservative scattering will only be considered in section 2.7). The phase function $p(\Theta)$

then obeys the normalization condition

$$\int \frac{\mathrm{d}\Omega'}{4\pi} p(\Theta) = \int_{-1}^{1} \frac{\mathrm{d}\cos\Theta}{2} p(\Theta) = 1. \tag{2.2}$$

The total cross section reads $\sigma = u^2/(4\pi)$, and the *scattering mean free path* ℓ is given by

$$\ell = \frac{1}{n\sigma} = \frac{4\pi}{nu^2} \tag{2.3}$$

where the density n of scatterers is assumed to be small, in such a way that we have $\ell \gg \lambda_0$.

In the particular case of isotropic scattering, the phase function $p(\Theta) = 1$ is a constant. In the general case of anisotropic scattering, the phase function $p(\Theta)$ is a non-trivial function of the scattering angle. As recalled in section 1, one has to distinguish between the scattering mean free path ℓ, which is the typical distance between two successive scattering events, and the *transport mean free path* ℓ^*, which represents the distance over which radiation loses memory of its direction. Both mean free paths are related by the following expression, well known from kinetic theory,

$$\tau^* = \frac{\ell^*}{\ell} = \frac{1}{1 - \langle\cos\Theta\rangle} \tag{2.4}$$

with

$$\langle\cos\Theta\rangle = \int_{-1}^{1} \frac{\mathrm{d}\cos\Theta}{2} \cos\Theta\, p(\Theta). \tag{2.5}$$

The dimensionless ratio τ^* will be referred to as the *anisotropy parameter*. We usually have $\tau^* \geqslant 1$, i.e. $\ell^* \geqslant \ell$, since the scattering cross section is often peaked in the forward direction, in a more or less prominent way. The regime of *very anisotropic scattering*, where $p(\Theta)$ is strongly peaked around the forward direction, corresponds to $\ell^* \gg \ell$. This regime will be investigated in detail in section 4.

2.1. General formalism

In this section we present some general formalism on anisotropic multiple scattering, thus extending the treatment of the isotropic case presented in [15]. Some of the results exposed below are already present in lecture notes by one of us [12].

We begin with a reminder of radiative transfer theory. In the regime $\ell \gg \lambda_0$ under consideration, in the absence of internal sources of radiation, and in stationary conditions, the quantity of interest is the specific intensity $I(\boldsymbol{r}, \boldsymbol{n})$ of radiation at the position $\boldsymbol{r}$, propagating in the direction $\boldsymbol{n}$. The specific intensity obeys the time-independent radiative transfer equation, that takes the following local form

$$\ell \boldsymbol{n} \cdot \nabla I(\boldsymbol{r}, \boldsymbol{n}) = \Gamma(\boldsymbol{r}, \boldsymbol{n}) - I(\boldsymbol{r}, \boldsymbol{n}). \tag{2.6}$$

The quantity

$$\Gamma(\boldsymbol{r}, \boldsymbol{n}) = \int \frac{\mathrm{d}\Omega'}{4\pi} p(\boldsymbol{n}, \boldsymbol{n}') I(\boldsymbol{r}, \boldsymbol{n}') \tag{2.7}$$

is commonly referred to as the source function. We recall that the phase function $p(\boldsymbol{n}, \boldsymbol{n}') = p(\Theta)$ only depends on the scattering angle Θ.

As recalled in section 1, the radiative transfer equation (2.6) [1–4] can be considered as a mesoscopic balance equation for the light intensity inside the diffusive medium, somewhat analogous to the Boltzmann equation in the kinetic theory of gases. It is equivalent to the Bethe–Salpeter equation, obtained by summing the ladder diagrams of the Born series expansion of the intensity Green's function of the problem. These diagrams are the dominant ones for $\ell \gg \lambda_0$, i.e. to leading order in the density n of scatterers.

We consider a sample of diffusive medium in the form of a slab of thickness L, limited by the two parallel planes $z = 0$ and $z = L$. The mean optical index n_0 of the slab can be different from the index n_1 of the surrounding medium. The index mismatch, measured by the ratio $m = n_0/n_1$, generates internal reflections at the interfaces. We introduce the optical depth $\tau = z/\ell$ of a point in the sample, and the optical thickness $b = L/\ell$ of the sample. Finally we use angular coordinates as in table 1: θ is the incidence angle, with the notation $\mu = \cos\theta$, while the azimuthal angle is denoted by φ.

Since the problem has rotational symmetry with respect to the z-axis, normal to the sample, and translational symmetry in the (x, y)-plane, it is natural to express the φ-dependence of the specific intensity and of the source function as Fourier series of the form

$$I(\boldsymbol{r}, \boldsymbol{n}) = \sum_{-\infty<m<+\infty} I^{(m)}(\tau, \mu)\mathrm{e}^{\mathrm{i}m\varphi} \qquad \Gamma(\boldsymbol{r}, \boldsymbol{n}) = \sum_{-\infty<m<+\infty} \Gamma^{(m)}(\tau, \mu)\mathrm{e}^{\mathrm{i}m\varphi} \tag{2.8}$$

where the integer m is the azimuthal number.

Furthermore, along the lines of [1, 4], for general anisotropic conservative scattering, we expand the phase function in Legendre polynomials as

$$p(\Theta) = \sum_{\ell\geqslant 0}(2\ell + 1)\varpi_\ell P_\ell(\cos\Theta). \tag{2.9}$$

We have $\varpi_0 = 1$ (see below), while the other coefficients ϖ_ℓ are only constrained by the positivity of the phase function.

In coordinates related to the sample, the phase function then reads

$$p(\boldsymbol{n}, \boldsymbol{n}') = p(\mu, \varphi, \mu', \varphi') = \sum_{-\infty<m<+\infty} p_m(\mu, \mu')\mathrm{e}^{\mathrm{i}m(\varphi-\varphi')} \tag{2.10}$$

with

$$p_m(\mu, \mu') = \sum_{\ell\geqslant|m|}(2\ell + 1)\varpi_\ell \frac{(\ell - |m|)!}{(\ell + |m|)!} P_{\ell,m}(\mu)P_{\ell,m}(\mu'). \tag{2.11}$$

The radiative transfer equation (2.6) thus reduces to

$$\mu\frac{\mathrm{d}}{\mathrm{d}\tau} I^{(m)}(\tau, \mu) = \Gamma^{(m)}(\tau, \mu) - I^{(m)}(\tau, \mu) \tag{2.12}$$

or equivalently

$$\mu\frac{\mathrm{d}}{\mathrm{d}\tau}[I^{(m)}(\tau, \mu)\mathrm{e}^{\tau/\mu}] = \Gamma^{(m)}(\tau, \mu)\mathrm{e}^{\tau/\mu} \tag{2.13}$$

and the source functions $\Gamma^{(m)}(\tau, \mu)$ are related to the specific intensities $I^{(m)}(\tau, \mu)$ through

$$\Gamma^{(m)}(\tau, \mu) = \mathcal{D}^{(m)}[I^{(m)}(\tau, \mu)] \tag{2.14}$$

where we have introduced the integral operators

$$\mathcal{D}^{(m)}[\Phi(\mu)] = \int_{-1}^{1} \frac{\mathrm{d}\mu'}{2} p_m(\mu, \mu')\Phi(\mu'). \tag{2.15}$$

We mention for further reference that the Legendre functions $\{P_{\ell,m}(\mu), \ell \geqslant |m|\}$ form a complete set of orthogonal eigenfunctions of the integral operator $\mathcal{D}^{(m)}$, with eigenvalues ϖ_ℓ, i.e.

$$\mathcal{D}^{(m)}[P_{\ell,m}(\mu)] = \varpi_\ell P_{\ell,m}(\mu). \tag{2.16}$$

Especially for $m = 0$ we have $P_{\ell,0}(\mu) = P_\ell(\mu)$, where the Legendre polynomials $P_\ell(\mu)$ already appeared in (2.9).

The following Legendre functions will be of special interest hereafter:

$$P_{0,0}(\mu) = P_0(\mu) = 1 \qquad P_{1,0}(\mu) = P_1(\mu) = \mu \qquad P_{1,1}(\mu) = \sqrt{1-\mu^2}. \tag{2.17}$$

Indeed the identity (2.16) has the following two special cases of interest.

- The eigenvalue $\varpi_0 = 1$, associated with the constant eigenfunction $P_0(\mu) = 1$ of $\mathcal{D}^{(0)}$, is a consequence of the conservative nature of the scattering mechanism, yielding diffusive behaviour in the long-distance limit. More explicitly,

$$\int_{-1}^{1} \frac{\mathrm{d}\mu'}{2} p_0(\mu, \mu') = 1. \tag{2.18}$$

- The first non-trivial eigenvalue ϖ_1 of both operators $\mathcal{D}^{(0)}$ and $\mathcal{D}^{(1)}$ is related to the anisotropy parameter τ^* of equation (2.4). Indeed, since $P_1(\mu) = \mu$ we have

$$\int_{-1}^{1} \frac{\mathrm{d}\mu'}{2} p_0(\mu, \mu')\mu' = \varpi_1 \mu \tag{2.19}$$

hence

$$\varpi_1 = \langle \cos\Theta \rangle = 1 - \frac{1}{\tau^*} \qquad \tau^* = \frac{1}{1-\varpi_1}. \tag{2.20}$$

We also notice that all the eigenvalues of the operators $\mathcal{D}^{(m)}$ are trivial in the particular case of isotropic scattering, since $\varpi_0 = 1$, while $\varpi_\ell = 0$ for $\ell \geqslant 1$.

2.2. Schwarzschild–Milne equation

We now turn to the derivation of the Schwarzschild–Milne equation in the general situation of anisotropic scattering. This key equation of radiative transfer theory will be the starting point of the following developments.

It turns out that most observables of interest can be derived by considering quantities with cylindrical symmetry around the z-axis, namely those corresponding to an azimuthal number $m = 0$. Henceforth we restrict the analysis to this sector, except in section 2.6, and we drop the superscript (0) for simplicity.

We consider first a half-space geometry ($b = \infty$). We assume that the limiting plane $\tau = 0$ of the sample is subjected to a cylindrically symmetric incident beam, characterized by an angle of incidence θ_a, i.e. that the incident intensity does not depend on the azimuthal angle φ_a. This is automatically satisfied at normal incidence ($\theta_a = 0$). Under these circumstances, the inward intensity on the limiting plane $\tau = 0^+$ contains both the normalized refracted incident beam, coming in a direction defined by μ_a, and the intensity coming from the bulk of the medium, after being reflected once at the interface. Using the radiative transfer equation (2.12), (2.13) for $m = 0$, we thus get

$$I(0^+, \mu, \mu_a) = 2\delta(\mu - \mu_a) + \frac{R(\mu)}{\mu} \int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-\tau/\mu} \Gamma(\tau, -\mu, \mu_a) \qquad (\mu > 0) \tag{2.21}$$

where the Fresnel reflection coefficient $R(\mu)$ is given in table 1. The only contribution to the outward intensity on this plane comes from the light rays which have experienced at least one scattering event, namely

$$I(0^+, -\mu, \mu_a) = \frac{1}{\mu} \int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-\tau/\mu} \Gamma(\tau, -\mu, \mu_a) \qquad (\mu > 0). \tag{2.22}$$

The radiative transfer equation (2.12), (2.13), together with the boundary conditions (2.21), (2.22) at $\tau = 0^+$, can then be recast in the integral form

$$I(\tau, \mu, \mu_a) = 2\delta(\mu - \mu_a)\mathrm{e}^{-\tau/\mu_a} + (K * \Gamma)(\tau, \mu, \mu_a). \qquad (2.23)$$

Here and throughout the following, the star denotes the convolution product

$$(K * \Gamma)(\tau, \mu, \mu_a) = \int_0^\infty \mathrm{d}\tau' \int_{-1}^1 \frac{\mathrm{d}\mu'}{2} K(\tau, \mu, \tau', \mu')\Gamma(\tau', \mu', \mu_a). \qquad (2.24)$$

The kernel K can be split into two components: $K = K_B + K_L$. The bulk kernel K_B contains the contributions to the intensity at depth τ arising from the scattering from either smaller or larger depths. The layer kernel K_L takes into account the intensity being scattered at depth τ' in the direction of the wall, then reflected there, and then scattered at depth τ. These kernels read explicitly

$$\begin{aligned} K_B(\tau, \mu, \tau', \mu') &= 2\delta(\mu - \mu')\theta(\mu(\tau - \tau'))\frac{1}{|\mu|}\mathrm{e}^{-(\tau-\tau')/\mu} \\ K_L(\tau, \mu, \tau', \mu') &= 2\delta(\mu + \mu')\theta(\mu)\frac{R(\mu)}{\mu}\mathrm{e}^{-(\tau+\tau')/\mu}. \end{aligned} \qquad (2.25)$$

The final step consists in using (2.14) in order to derive from (2.23) the following closed integral equation for the source function

$$\Gamma(\tau, \mu, \mu_a) = p_0(\mu, \mu_a)\mathrm{e}^{-\tau/\mu_a} + (M * \Gamma)(\tau, \mu, \mu_a) \qquad (2.26)$$

which we refer to as the Schwarzschild–Milne equation of the problem.

The kernel M has the following two components, in analogy with the kernel K from which it derives

$$\begin{aligned} M_B(\tau, \mu, \tau', \mu') &= \theta(\mu'(\tau - \tau'))\frac{p_0(\mu, \mu')}{|\mu'|}\mathrm{e}^{-(\tau-\tau')/\mu'} \\ M_L(\tau, \mu, \tau', \mu') &= \theta(-\mu')\frac{p_0(\mu, -\mu')}{|\mu'|}R(-\mu')\mathrm{e}^{(\tau+\tau')/\mu'} \end{aligned} \qquad (2.27)$$

so that (2.26) reads explicitly

$$\begin{aligned} \Gamma(\tau, \mu, \mu_a) = p_0(\mu, \mu_a)\mathrm{e}^{-\tau/\mu_a} &+ \int_0^\tau \mathrm{d}\tau' \int_0^1 \frac{\mathrm{d}\mu'}{2\mu'} p_0(\mu, \mu')\mathrm{e}^{-(\tau-\tau')/\mu'}\Gamma(\tau', \mu', \mu_a) \\ &+ \int_\tau^\infty \mathrm{d}\tau' \int_0^1 \frac{\mathrm{d}\mu'}{2\mu'} p_0(\mu, -\mu')\mathrm{e}^{-(\tau'-\tau)/\mu'}\Gamma(\tau', -\mu', \mu_a) \\ &+ \int_0^\infty \mathrm{d}\tau' \int_0^1 \frac{\mathrm{d}\mu'}{2\mu'} R(\mu')p_0(\mu, \mu')\mathrm{e}^{-(\tau+\tau')/\mu'}\Gamma(\tau', -\mu', \mu_a). \end{aligned} \qquad (2.28)$$

In the case of isotropic scattering, the phase function $p_0(\mu, \mu') = 1$ is a constant, and the source function $\Gamma(\tau, \mu, \mu_a)$ does not depend on μ. The Schwarzschild–Milne equation (2.26), (2.28) thus takes a simpler form, that has been extensively studied in [15]. The rest of this section is devoted to an extension of the results derived there to the general case of anisotropic scattering.

2.3. Solutions of Schwarzschild–Milne equation and sum rules

In the general case of conservative anisotropic scattering, the Schwarzschild–Milne equation (2.26), (2.28) has a special solution $\Gamma_S(\tau, \mu, \mu_a)$ which remains bounded as $\tau \to \infty$, whereas the associated homogeneous equation has a linearly growing solution

$\Gamma_H(\tau, \mu)$. More precisely, it can be checked by means of (2.18), (2.19) that both source functions and the associated specific intensities have the following asymptotic behaviour for large depths ($\tau \gg 1$), up to exponentially small corrections

$$\begin{aligned}
&\Gamma_S(\tau, \mu, \mu_a) \approx \tau_1(\mu_a) \\
&I_S(\tau, \mu, \mu_a) \approx \tau_1(\mu_a) \\
&\Gamma_H(\tau, \mu) \approx \tau - \mu(\tau^* - 1) + \tau_0\tau^* \\
&I_H(\tau, \mu) \approx \tau - \mu\tau^* + \tau_0\tau^*.
\end{aligned} \tag{2.29}$$

The quantities τ_0 and $\tau_1(\mu_a)$ are unknown so far. Let us anticipate that they play a central role in the following, in the sense that they will bear the full non-trivial dependence of quantities on the scattering mechanism. These quantities also obey two groups of sum rules, (2.37), (2.38) and (2.41), (2.42), to be derived below.

To do so, it is most convenient to introduce the Green's function $G_S(\tau, \mu, \tau', \mu')$ of the problem, along the lines of [15]. It is defined as the solution which remains bounded as $\tau \to \infty$ of the equation

$$G_S(\tau, \mu, \tau', \mu') = p_0(\mu, \mu')\delta(\tau - \tau') + (M * G_S)(\tau, \mu, \tau', \mu'). \tag{2.30}$$

The kernel K and the Green's function G_S possess the symmetry properties

$$\begin{aligned}
&K(\tau, \mu, \tau', \mu') = K(\tau', -\mu', \tau, -\mu) \\
&G_S(\tau, \mu, \tau', \mu') = G_S(\tau', -\mu', \tau, -\mu)
\end{aligned} \tag{2.31}$$

which merely express the time-reversal symmetry of any sequence of scattering events.

As a consequence of (2.26), the special solution $\Gamma_S(\tau, \mu, \mu_a)$ can be expressed in terms of the Green's function as

$$\Gamma_S(\tau, \mu, \mu_a) = \int_0^\infty \mathrm{d}\tau'\, \mathrm{e}^{-\tau'/\mu_a} G_S(\tau, \mu, \tau', \mu_a). \tag{2.32}$$

We also define for further reference the following bistatic coefficient

$$\begin{aligned}
\gamma(\mu_a, \mu_b) &= \int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-\tau/\mu_b} \Gamma_S(\tau, -\mu_b, \mu_a) \\
&= \int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-\tau/\mu_b} \int_0^\infty \mathrm{d}\tau'\, \mathrm{e}^{-\tau'/\mu_a} G_S(\tau, -\mu_b, \tau', \mu_a).
\end{aligned} \tag{2.33}$$

The latter expression defines the bistatic coefficient for any complex values of its arguments with $\mathrm{Re}\,\mu_a > 0$, $\mathrm{Re}\,\mu_b > 0$, even outside the physical range $\mu_a, \mu_b \leqslant 1$. The symmetry $\gamma(\mu_a, \mu_b) = \gamma(\mu_b, \mu_a)$ is a consequence of the properties (2.31). It is thus again due to time-reversal symmetry.

On the other hand, a relationship between both solutions $\Gamma_H(\tau, \mu)$ and $\Gamma_S(\tau, \mu, \mu_a)$ of the Schwarzschild–Milne equation can be derived as follows. The Green's function $G_S(\tau, \mu, \tau', \mu')$ is clearly asymptotically proportional to the homogeneous solution $\Gamma_H(\tau, \mu)$ when τ' goes to infinity, namely

$$\lim_{\tau' \to \infty} G_S(\tau, \mu, \tau', \mu') = \frac{1}{D} \Gamma_H(\tau, \mu) \tag{2.34}$$

where the proportionality constant D will be shown in a while to be equal to the dimensionless diffusion coefficient (2.39).

As a consequence of equations (2.29), (2.32)–(2.34), we have

$$\tau_1(\mu_a) = \lim_{\mu_b \to \infty} \frac{\gamma(\mu_a, \mu_b)}{\mu_b} = \frac{1}{D} \int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-\tau/\mu_a} \Gamma_H(\tau, -\mu_a). \tag{2.35}$$

We now turn to the actual derivation of the sum rules obeyed by the quantities defined so far, which are related to the so-called F and K-integrals, with the notations of Chandrasekhar [1].

The first group of two sum rules is a consequence of the conservation of the flux in the z-direction in a non-absorbing medium, given by the following F-integral

$$F(\tau) = \int_{-1}^{1} \frac{\mathrm{d}\mu}{2} \mu I(\tau, \mu). \tag{2.36}$$

It can indeed be checked, using equation (2.12), that $\mathrm{d}F/\mathrm{d}\tau = 0$.

We investigate first the F-integral F_S associated with the special solution $\Gamma_S(\tau, \mu, \mu_a)$. Considering the $\tau \to \infty$ limit determines $F_S = 0$, whereas considering the $\tau \to 0$ limit yields the sum rule

$$\int_{0}^{1} \frac{\mathrm{d}\mu}{2} T(\mu)\gamma(\mu, \mu_a) = \mu_a \tag{2.37}$$

where the transmission coefficient $T(\mu) = 1 - R(\mu)$ is given in table 1. The $\mu_a \to \infty$ limit of (2.37), together with (2.35), yields another sum rule, namely

$$\int_{0}^{1} \frac{\mathrm{d}\mu}{2} T(\mu)\tau_1(\mu) = 1. \tag{2.38}$$

We can evaluate in a similar way the F-integral F_H associated with the homogeneous solution $\Gamma_H(\tau, \mu)$. This yields no independent sum rule, but leads to the identification of the constant D of (2.30) with the dimensionless diffusion coefficient

$$D = \frac{\tau^*}{3}. \tag{2.39}$$

The diffusion coefficient indeed reads $D_{\mathrm{phys}} = c\ell D = c\ell^*/3$ in physical units [1–4]. The transport velocity indeed coincides with the velocity of light in vacuum c, to leading order in the regime $\ell \gg \lambda_0$.

Besides the sum rules (2.37), (2.38), which were already given in [15] for isotropic scattering, the radiative transfer equation also admits another group of two sum rules, which are novel in this context, and whose intuitive interpretation is less evident. Consider the so-called K-integral, again with the notations of Chandrasekhar [1]

$$K(\tau) = \int_{-1}^{1} \frac{\mathrm{d}\mu}{2} \mu^2 I(\tau, \mu). \tag{2.40}$$

It can be checked that equations (2.12), (2.18)–(2.20) yield $\mathrm{d}K/\mathrm{d}\tau = -F/\tau^*$, hence $K(\tau) = -F\tau/\tau^* + K_0$, with K_0 being independent of τ. By considering the K-integrals associated with the special solution $\Gamma_S(\tau, \mu, \mu_a)$ and with the homogeneous solution $\Gamma_H(\tau, \mu)$, we obtain after some algebra the following sum rules

$$\int_{0}^{1} \frac{\mathrm{d}\mu}{2} (1 + R(\mu))\mu\gamma(\mu, \mu_a) = \frac{\tau_1(\mu_a)}{3} - \mu_a^2 \tag{2.41}$$

$$\int_{0}^{1} \frac{\mathrm{d}\mu}{2} (1 + R(\mu))\mu\tau_1(\mu) = \tau_0. \tag{2.42}$$

2.4. Diffuse reflected intensity

The evaluation of the angle-resolved diffuse reflected intensity by means of the general formalism exposed above closely follows the lines of [4, 15]. We consider a half-space geometry, and we assume that the limiting plane of the sample is subjected to a cylindrically symmetric incident beam, characterized by an angle of incidence θ_a. This technical assumption includes the case of a plane wave at normal incidence ($\theta_a = 0$). Under these circumstances, the diffuse reflected intensity per solid angle $\mathrm{d}\Omega_b$ reads

$$\frac{\mathrm{d}R(a \to b)}{\mathrm{d}\Omega_b} = A^R(\theta_a, \theta_b) = \frac{\cos\theta_a}{4\pi m^2} \frac{T_a T_b}{\mu_a \mu_b} \gamma(\mu_a, \mu_b) \tag{2.43}$$

where we have again used the notations of table 1: $m = n_0/n_1$ is the index mismatch, and $T_a = T(\mu_a)$ and $T_b = T(\mu_b)$ are the transmission coefficients in the incident and outgoing directions, respectively.

The essential factor in the result (2.43) is the bistatic coefficient $\gamma(\mu_a, \mu_b)$, whose definition and general properties have been exposed in section 2.3. It will be evaluated more explicitly in the large index mismatch regime in section 3, and in the very anisotropic regime in section 4.

2.5. Diffuse transmitted intensity

In this section we consider the angle-resolved mean transmission of an optically thick slab, of thickness $L = b\ell$, with $b \gg 1$ being large but finite. A generalization of the reasoning of section 2.1 allows us to write down the following Schwarzschild–Milne equation in this geometry

$$\begin{aligned}
\Gamma_b(\tau, \mu, \mu_a) = {} & p_0(\mu, \mu_a)\mathrm{e}^{-\tau/\mu_a} + \int_0^\tau \mathrm{d}\tau' \int_0^1 \frac{\mathrm{d}\mu'}{2\mu'} p_0(\mu, \mu')\mathrm{e}^{-(\tau-\tau')/\mu'} \Gamma_b(\tau', \mu', \mu_a) \\
& + \int_\tau^b \mathrm{d}\tau' \int_0^1 \frac{\mathrm{d}\mu'}{2\mu'} p_0(\mu, -\mu')\mathrm{e}^{-(\tau'-\tau)/\mu'} \Gamma_b(\tau', -\mu', \mu_a) \\
& + \int_0^b \mathrm{d}\tau' \int_0^1 \frac{\mathrm{d}\mu'}{2\mu'} R(\mu') p_0(\mu, \mu')\mathrm{e}^{-(\tau+\tau')/\mu'} \Gamma_b(\tau', -\mu', \mu_a) \\
& + \int_0^b \mathrm{d}\tau' \int_0^1 \frac{\mathrm{d}\mu'}{2\mu'} R(-\mu') p_0(\mu, -\mu')\mathrm{e}^{-(2b-\tau-\tau')/\mu'} \Gamma_b(\tau', \mu', \mu_a).
\end{aligned} \tag{2.44}$$

The solution of this equation for a thick slab can be constructed from both solutions Γ_S and Γ_H of the half-space geometry, by means of a matching procedure, along the lines of [4, 15]. Using the asymptotic forms (2.29), we obtain

$$\Gamma_b(\tau, \mu, \mu_a) \approx \begin{cases} \Gamma_S(\tau, \mu, \mu_a) - \dfrac{\tau_1(\mu_a)}{b + 2\tau_0\tau^*} \Gamma_H(\tau, \mu) & (\tau \text{ finite}, b - \tau \gg 1) \\ \dfrac{\tau_1(\mu_a)}{b + 2\tau_0\tau^*} \Gamma_H(b - \tau, -\mu) & (b - \tau \text{ finite}, \tau \gg 1). \end{cases} \tag{2.45}$$

Both expressions lead to a linear (diffusive) behaviour [3, 15] in the bulk of the sample ($\tau \gg 1$, $b - \tau \gg 1$), namely

$$\Gamma_b(\tau, \mu, \mu_a) = \frac{\tau_1(\mu_a)}{b + 2\tau_0\tau^*}[b - \tau + \tau_0\tau^* + \mu(\tau^* - 1)] + \mathcal{O}(\mathrm{e}^{-\tau}, \mathrm{e}^{-(b-\tau)}). \tag{2.46}$$

The derivation of the diffuse transmitted intensity per solid angle element $\mathrm{d}\Omega_b$ again closely follows the lines of [4, 15]. We obtain

$$\frac{\mathrm{d}T(a \to b)}{\mathrm{d}\Omega_b} = \frac{\tau^*}{b + 2\tau_0\tau^*} A^T(\theta_a, \theta_b) = \frac{\ell^*}{L + 2\tau_0\ell^*} A^T(\theta_a, \theta_b) \tag{2.47}$$

with

$$A^T(\theta_a, \theta_b) = \frac{\cos\theta_a}{12\pi m^2} \frac{T_a T_b}{\mu_a \mu_b} \tau_1(\mu_a)\tau_1(\mu_b) \tag{2.48}$$

where we have again used the notations of table 1: $m = n_0/n_1$ is the index mismatch, and $T_a = T(\mu_a)$ and $T_b = T(\mu_b)$ are the transmission coefficients in the incident and outgoing directions, respectively.

The essential ingredient in the above result is the function $\tau_1(\mu)$, whose definition and general properties have been exposed in section 2.3. It will also be evaluated more explicitly in the large index mismatch regime in section 3, and in the very anisotropic regime in section 4.

The result (2.47) shows that the effective thickness of the sample is $(b + 2\tau_0\tau^*)\ell = L + 2\tau_0\ell^*$. In other words, $z_0 = \tau_0\ell^*$ represents the thickness of a skin layer. This quantity is also referred to as the injection depth, or the extrapolation length of the problem.

2.6. Enhanced backscattering cone

The general formalism exposed above can be extended to the study of the enhanced backscattering phenomenon, which takes place in a narrow cone in the vicinity of the exact backscattering direction [5]. This phenomenon is due to the constructive interference between any path in the medium and its time-reversed counterpart. One of the goals of this section is to derive a quantitative estimate of the width of the cone, with emphasis on its dependence on the anisotropy of the scattering mechanism. As recalled in section 1, the form of the backscattering cone can be predicted by summing the so-called maximally crossed diagrams [10, 11]. This can be performed by means of a careful treatment of radiative transfer theory [10–12, 15, 16].

We restrict the analysis to normal incidence ($\theta_a = 0$), and to the geometry of a half-space diffusive medium. We introduce the dimensionless transverse wavevector

$$\boldsymbol{Q} = \boldsymbol{q}\ell \tag{2.49}$$

and its magnitude

$$Q = q\ell = k_0\ell\theta' = k_1\ell\theta > 0 \tag{2.50}$$

where θ' and θ are the incidence angles of the outgoing radiation, and k_0 and k_1 are its wavenumbers, respectively inside and outside the diffusive medium, according to table 1.

Along the lines of [10–12, 15, 16], the reflected intensity in the vicinity of the backscattering direction, i.e. for $\theta \ll 1$, $k_1\ell \gg 1$, and Q fixed, takes the form

$$A^C(Q) \approx \frac{T(1)^2}{4\pi m^2}[\gamma(1, 1) + \gamma_C(Q) - p_0(1, -1)/2] \tag{2.51}$$

where

- the sum of ladder diagrams, $\gamma(1, 1)$, yields the background reflected intensity in the normal direction, in agreement with equation (2.43);
- the sum of the maximally crossed diagrams, $\gamma_C(Q)$, represents the contribution of the interference between the sequences of any number ($n \geqslant 1$) of scattering events and their time-reversed counterparts;

- the subtracted third term is the contribution of the single-scattering events ($n = 1$), which are invariant under time inversion, and must not be counted twice.

We define the enhancement factor

$$B(Q) = \frac{A^C(Q)}{A^R(0,0)} = 1 + \frac{\gamma_C(Q) - p_0(1,-1)/2}{\gamma(1,1)}. \tag{2.52}$$

It turns out that the peak value of the interference contribution coincides with the background term, i.e. $\gamma_C(0) = \gamma(1,1)$. Hence the enhancement factor at the top of the cone, namely

$$B(0) = 2 - \frac{p_0(1,-1)}{2\gamma(1,1)} \tag{2.53}$$

nearly equals two, up to the small contribution of single-scattering events.

We now turn to the actual determination of $\gamma_C(Q)$ [10–12, 15, 16]. Basically, the transverse wavevector $\boldsymbol{Q}$ causes a dephasing which amounts to replacing the pure exponential damping $\exp(-\tau/\mu)$ of unscattered intensity by the complex exponential $\exp(-(1 - \mathrm{i}\boldsymbol{Q} \cdot \boldsymbol{n})\tau/\mu)$. Because of the vector nature of $\boldsymbol{Q}$, the source functions $\Gamma^{(m)}(Q, \tau, \mu)$ pertaining to all sectors defined by the azimuthal integer m are coupled to each other.

We choose coordinates such that $\boldsymbol{Q}$ is oriented along the positive y-axis, in order to simplify notations. The source functions then obey coupled Q-dependent Schwarzschild–Milne equations of the form

$$\Gamma^{(m)}(Q, \tau, \mu) = \delta_{m,0}\, p_0(\mu, 1)\mathrm{e}^{-\tau} + \sum_{-\infty<k<+\infty} (M^{(m,k)} * \Gamma^{(k)})(Q, \tau, \mu) \tag{2.54}$$

generalizing equation (2.26). The Q-dependent Schwarzschild–Milne kernels read $M^{(m,k)} = M_B^{(m,k)} + M_L^{(m,k)}$, with

$$\begin{aligned} M_B^{(m,k)}(Q, \tau, \mu, \tau', \mu') &= \theta(\mu'(\tau - \tau'))\frac{p_m(\mu, \mu')}{|\mu'|}\mathrm{e}^{-(\tau-\tau')/\mu'} J_{m-k}\left(Q\frac{\tau - \tau'}{\mu'}\sqrt{1-\mu'^2}\right) \\ M_L^{(m,k)}(Q, \tau, \mu, \tau', \mu') &= \theta(-\mu')\frac{p_m(\mu, -\mu')}{|\mu'|}R(-\mu')\mathrm{e}^{(\tau+\tau')/\mu'} J_{m-k}\left(Q\frac{\tau + \tau'}{|\mu'|}\sqrt{1-\mu'^2}\right) \end{aligned} \tag{2.55}$$

where $J_m(z)$ denotes the Bessel function of integer order. The source functions have the following property

$$\Gamma^{(-m)}(Q, \tau, \mu) = (-1)^m \Gamma^{(m)}(Q, \tau, \mu) \tag{2.56}$$

which they inherit from an analogous symmetry property of the Bessel functions, i.e. $J_{-m}(z) = (-1)^m J_m(z)$,

Finally, the shape of the backscattering cone is given by

$$\gamma_C(Q) = \int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-\tau}\Gamma^{(0)}(Q, \tau, -1). \tag{2.57}$$

The top of the backscattering cone, described by the small-Q behaviour of $\gamma_C(Q)$, is of special interest, especially because of its universality. Indeed it is due to the contribution of long paths in the diffusive medium, along which the radiation undergoes many scattering events. In contrast, the wings of the cone, corresponding to a large reduced wavevector Q, only involve short sequences of two, three, etc scattering events, and are therefore expected to depend on the details of the scattering mechanism. This is already apparent in the subtracted term in equations (2.52), (2.53), which involves the single-scattering cross section in the direction of exact backscattering.

The universal small-Q behaviour of the backscattering cone can be determined as follows. We look for an approximate solution for $Q \ll 1$ to the coupled Schwarzschild–Milne equations (2.54) as a decaying exponential, with an inverse extinction length s_0, times expansions in Legendre functions of the form

$$\Gamma^{(m)}(Q,\tau,\mu) = \mathrm{e}^{-s_0\tau} G^{(m)}(Q,\mu) \qquad \text{with } G^{(m)}(Q,\mu) = \sum_{\ell \geqslant |m|} c_{\ell,m} P_{\ell,m}(\mu). \tag{2.58}$$

First, we observe that the arguments of the Bessel functions in the kernels (2.55) are proportional to Q. Since we have $J_m(z) \approx (z/2)^m/m!$ for small z and $m \geqslant 0$, we therefore expect that the coefficients of the expansion (2.58) fall off as $c_{\ell,m} \sim Q^{|m|}$. By virtue of the symmetry (2.56), we can thus restrict the analysis to the sectors $m = 0$ and $m = 1$. Second, we make the hypothesis, to be checked later on, that the inverse extinction length s_0 is proportional to Q. Then, deep in the bulk of the medium, i.e. for $\tau \gg 1$, the integral equations (2.54) can be approximated by differential equations, obtained by expanding the source functions in powers of $(\tau'-\tau)$, keeping only the first two derivatives, and consistently the first two powers of Q. We thus obtain the following two coupled equations

$$\begin{aligned}\Gamma^{(0)}(Q,\tau,\mu) = \mathcal{D}^{(0)}\bigg[&\Gamma^{(0)}(Q,\tau,\mu) - \mu\frac{\mathrm{d}}{\mathrm{d}\tau}\Gamma^{(0)}(Q,\tau,\mu) + \mu^2\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\Gamma^{(0)}(Q,\tau,\mu)\\ &-\frac{Q^2}{2}(1-\mu^2)\Gamma^{(0)}(Q,\tau,\mu) + Q\sqrt{1-\mu^2}\,\Gamma^{(1)}(Q,\tau,\mu) + \cdots\bigg]\end{aligned} \tag{2.59a}$$

$$\Gamma^{(1)}(Q,\tau,\mu) = \mathcal{D}^{(1)}\left[\Gamma^{(1)}(Q,\tau,\mu) - \frac{Q}{2}\sqrt{1-\mu^2}\,\Gamma^{(0)}(Q,\tau,\mu) + \cdots\right]. \tag{2.59b}$$

We first solve (2.59*b*) as follows. Since we only need a leading order estimate of $\Gamma^{(1)}(Q,\tau,\mu)$, we only keep the first coefficient $c_{1,1}$ of the expansion (2.58). Using (2.16), (2.17), we thus obtain

$$c_{1,1} \approx -\frac{Q}{2}(\tau^* - 1)c_{0,0}. \tag{2.60}$$

By inserting this last result into (2.59*a*), and making use of (2.16) and (A.4), we obtain the following recursion relations for the coefficients $c_{\ell,0}$

$$\begin{aligned}\frac{c_{\ell,0}}{\varpi_\ell} \approx \left(1 - \frac{Q^2\tau^*}{2}\right)c_{\ell,0} &+ s_0\left(\frac{\ell}{2\ell-1}c_{\ell-1,0} + \frac{\ell+1}{2\ell+3}c_{\ell+1,0}\right) + \left(s_0^2 + \frac{Q^2\tau^*}{2}\right)\\ &\times\left(\frac{\ell(\ell-1)}{(2\ell-1)(2\ell-3)}c_{\ell-2,0} + \frac{2\ell^2+2\ell-1}{(2\ell-1)(2\ell+3)}c_{\ell,0}\right.\\ &\left.+\frac{(\ell+1)(\ell+2)}{(2\ell+3)(2\ell+5)}c_{\ell+2,0}\right).\end{aligned} \tag{2.61}$$

It is clear from the structure of these relations that, when Q is small, the coefficients $c_{\ell,0} \sim Q^\ell$ decay rapidly. Keeping this hierarchy in mind, and using $\varpi_0 = 1$, we obtain the estimates

$$c_{1,0} \approx Q(\tau^* - 1)c_{0,0} \tag{2.62}$$

and

$$s_0 \approx Q. \tag{2.63}$$

This last result corroborates the hypothesis made in the derivation of (2.59). We shall come back to its meaning at the end of section 2.7.

The next step also follows the lines of [15]. The small-Q behaviour of $\Gamma_C(Q,\tau,\mu)$ has a term proportional to Q, which is proportional to the homogeneous solution $\Gamma_H(\tau,\mu)$ of the Schwarzschild–Milne equation (2.26), (2.28). Indeed, consider the right-hand side of (2.54) for $m=0$. The leading Q-dependence there comes either from the action of $M^{(0,0)}$ on $\Gamma^{(0)}$, or from the action of $M^{(0,\pm 1)}$ on $\Gamma^{(\pm 1)}$. All these explicit Q-dependences begin with Q^2. Putting everything together, we are left with the following estimates of the source function $\Gamma^{(0)}(Q,\tau,\mu)$. For $Q \ll 1$ and fixed τ we have

$$\Gamma^{(0)}(Q,\tau,\mu) = \Gamma_S(\tau,\mu,1) - Q\tau_1(1)\Gamma_H(\tau,\mu) + \mathcal{O}(Q^2) \tag{2.64}$$

whereas for $Q \ll 1$ and $\tau \gg 1$ simultaneously we get

$$\Gamma^{(0)}(Q,\tau,\mu) = \tau_1(1)\mathrm{e}^{-Q\tau}(1 + Q(\mu(\tau^* - 1) - \tau_0\tau^*) + \mathcal{O}(Q^2)). \tag{2.65}$$

The universal peak of the backscattering cone is then evaluated by inserting the estimate (2.64) into (2.57), using (2.33), (2.35). We thus obtain the following expression

$$\gamma_C(Q) = \gamma(1,1)\left(1 - \frac{Q}{\Delta Q} + \mathcal{O}(Q^2)\right) \tag{2.66}$$

where the width of the cone reads

$$\Delta Q = \frac{3\gamma(1,1)}{\tau_1(1)^2\tau^*} \tag{2.67}$$

i.e. in physical units

$$\Delta\theta = \frac{3\gamma(1,1)}{\tau_1(1)^2}\frac{1}{k_1\ell^*} \tag{2.68}$$

with k_1 being the wavenumber of radiation outside the diffusive medium. This simple $1/\ell^*$ law is already predicted by the diffusion approximation [31, 32].

2.7. Extinction and absorption lengths

Up to now we have assumed that the diffusive medium is conservative. This means that the light only experiences elastic collisions; there is neither absorption, nor inelastic scattering, implying the normalization (2.2) of the phase function. We now want to discuss briefly the case of a weakly absorbing diffusive medium, characterized by a non-trivial albedo a such that $1-a \ll 1$. In this case the diffuse intensity is expected to die-off exponentially inside the medium, with a characteristic absorption length L_{abs}.

The known expression [2, 3, 12] of the absorption length can easily be recovered by means of the formalism exposed in the previous section, in the case of general anisotropic scattering and in the regime of weak absorption. We shall actually determine the extinction length of the more general problem defined by the coupled Q-dependent Schwarzschild–Milne equations (2.54). We look for slowly varying source functions of the form (2.58), along the lines of section 2.6. The main difference is that we have now $\varpi_0 = a \neq 1$. It turns out that the estimates (2.60), (2.62) of the amplitudes $c_{1,m}$ still hold, whereas we obtain

$$s_0^2 \approx Q^2 + \frac{3(1-a)}{\tau^*}. \tag{2.69}$$

The Q-dependent extinction length in the presence of absorption therefore reads

$$L_{\mathrm{ext}}(Q) = \frac{\ell}{s_0} \approx \frac{\ell}{\left(Q^2 + \frac{3(1-a)}{\tau^*}\right)^{1/2}} \qquad (Q \ll 1, 1-a \ll 1). \tag{2.70}$$

The usual absorption length is obtained by setting $Q = 0$ in the above expression, namely

$$L_{\rm abs} \approx \left(\frac{\ell\ell^*}{3(1-a)}\right)^{1/2} \qquad (1 - a \ll 1) \tag{2.71}$$

in agreement with [2, 3, 12]. Another particular case is conservative scattering, in the absence of absorption, where we recover the result (2.63), namely

$$L_{\rm ext}(Q) \approx \frac{\ell}{Q} \approx \frac{1}{q}. \tag{2.72}$$

This simple result holds for a general anisotropic scattering. It is a manifestation of the isotropic character of the long-distance diffusive behaviour of the multiple scattering problem. We shall also come back to this point in section 5.

3. Large index mismatch regime

In this section, we extend to the general case of anisotropic scattering the approach of [15], which predicts the behaviour of quantities in the regime where the optical indices n_0 and n_1 of the diffusive medium and of the surroundings are very different from each other, i.e. when their ratio $m = n_0/n_1$ is either very small or very large. As already pointed out in [15], important simplifications occur in these regimes of a large index mismatch, where the Fresnel transmission coefficient of the boundaries of the medium is very small. To be more specific, radiation cannot enter the medium (respectively, leave the medium) in the limit $m \ll 1$ (respectively, $m \gg 1$), except at normal incidence. Reference [12] already contains part of the results of this section.

3.1. Diffuse reflection and transmission

Along the lines of [15], we evaluate the reflected and transmitted intensities in the large index mismatch regime by means of the following singular perturbative expansion of the Green's function $G_S(\tau, \mu, \tau', \mu')$, defined by (2.30).

The starting point consists in noticing the identity

$$M_L(\tau, \mu, \tau', \mu') = R(-\mu')M_B(\tau, \mu, -\tau', -\mu') \tag{3.1}$$

between both kernels defined in (2.27). Using $R(\mu) = 1 - T(\mu)$, we can recast (2.30) as

$$\begin{aligned} G_S(\tau, \mu, \tau', \mu') &= p_0(\mu, \mu')\delta(\tau - \tau') \\ &\quad + \int_0^\infty \mathrm{d}\tau'' \int_{-1}^1 \frac{\mathrm{d}\mu''}{2} [M_B(\tau - \tau'', \mu, 0, \mu'') + M_B(\tau + \tau'', \mu, 0, -\mu'')] \\ &\quad \times G_S(\tau'', \mu'', \tau', \mu') \\ &\quad - \int_0^\infty \mathrm{d}\tau'' \int_0^1 \frac{\mathrm{d}\mu''}{2} T(\mu'')M_B(\tau, \mu, -\tau'', -\mu'')G_S(\tau'', \mu'', \tau', \mu'). \end{aligned} \tag{3.2}$$

In the limit of an infinitely strong index mismatch, i.e. for $m = 0$ or $m = \infty$, the transmission coefficient $T(\mu)$ vanishes identically, so that the last integral of equation (3.2), involving $T(\mu)$, is absent. It can be checked that the remaining terms only determine the Green's function up to an additive constant. This constant is only fixed by the action of the last integral, involving $T(\mu)$, in (3.2). It can therefore be expected to diverge as $m \to 0$ and $m \to \infty$.

In order to demonstrate this explicitly, we expand the Green's function according to

$$G_S(\tau, \mu, \tau', \mu') = C_S + G_0(\tau, \mu, \tau', \mu') + \cdots \tag{3.3}$$

with the hypothesis that C_S diverges, whereas $G_0(\tau, \mu, \tau', \mu')$ remains finite, and the dots stand for terms which go to zero, as $m \to 0$ or $m \to \infty$. The *finite part* $G_0(\tau, \mu, \tau', \mu')$ obeys the following equation

$$\begin{aligned} G_0(\tau, \mu, \tau', \mu') &= p_0(\mu, \mu')\delta(\tau - \tau') \\ &\quad + \int_0^\infty \mathrm{d}\tau'' \int_{-1}^1 \frac{\mathrm{d}\mu''}{2}[M_B(\tau - \tau'', \mu, 0, \mu'') + M_B(\tau + \tau'', \mu, 0, -\mu'')] \\ &\quad \times G_0(\tau'', \mu'', \tau', \mu') - C_S \int_0^\infty \mathrm{d}\tau'' \int_0^1 \frac{\mathrm{d}\mu''}{2} T(\mu'') M_B(\tau, \mu, -\tau'', -\mu'') \end{aligned} \tag{3.4}$$

together with the consistency condition

$$\int_0^\infty \mathrm{d}\tau \int_0^\infty \mathrm{d}\tau' \int_{-1}^1 \frac{\mathrm{d}\mu}{2} \int_0^1 \frac{\mathrm{d}\mu'}{2} T(\mu') M_B(\tau + \tau', \mu, 0, -\mu') G_0(\tau', \mu', \tau'', \mu'') = 0 \tag{3.5}$$

derived along the lines of [15].

The constant C_S of the expansion (3.3) can be derived by integrating equation (3.4) over the variables $0 < \tau < \infty$ and $-1 < \mu < 1$. This yields

$$C_S = \frac{4}{\mathcal{T}} \tag{3.6}$$

where $\mathcal{T}$ is the *mean flux transmission coefficient*

$$\mathcal{T} = 2\int_0^1 \mu T(\mu)\,\mathrm{d}\mu = \begin{cases} \dfrac{4m(m+2)}{3(m+1)^2} & (m \leqslant 1) \\ \dfrac{4(2m+1)}{3m^2(m+1)^2} & (m \geqslant 1). \end{cases} \tag{3.7}$$

This quantity assumes its maximum $\mathcal{T} = 1$ in the absence of any index mismatch, i.e. for $m = 1$, and it vanishes in both cases of a large index mismatch, according to

$$\mathcal{T} \approx \begin{cases} \dfrac{8m}{3} & (m \ll 1) \\ \dfrac{8}{3m^3} & (m \gg 1). \end{cases} \tag{3.8}$$

The asymptotic behaviour in the limits $m \ll 1$ and $m \gg 1$ of the quantities pertaining to reflection and transmission is immediately obtained by replacing in equations (2.33), (2.35) the Green's function $G_S(\tau, \mu, \tau', \mu')$ by its leading constant term C_S. We thus obtain

$$\tau_0 \approx \frac{4}{3\mathcal{T}} \qquad \tau_1(\mu) \approx \frac{4\mu}{\mathcal{T}} \qquad \gamma(\mu_a, \mu_b) \approx \frac{4\mu_a\mu_b}{\mathcal{T}}. \tag{3.9}$$

These predictions are identical to those derived in [15], in the case of isotropic scattering. We have thus shown that the quantities which determine the diffuse reflected and transmitted intensities do not depend *at all* on the anisotropy of the cross section in the large index mismatch limit.

3.2. Enhanced backscattering cone

The shape $\gamma_C(Q)$ of the cone of enhanced backscattering for a normal incidence can also be evaluated analytically in the regimes $m \ll 1$ or $m \gg 1$ of a large index mismatch. By inserting the estimates (3.9) into the general result (2.67), we find that the width of the cone is small in the large index mismatch regime, since it is proportional to the mean transmission

$\mathcal{T}$. This observation suggests that we should consider the scaling regime where both Q and $\mathcal{T}$ are simultaneously small.

In order to investigate this regime, we first recast the Q-dependent coupled Schwarzschild-Milne equations (2.54) in a form similar to equation (3.2). We then look for a solution of the form (2.58), where the inverse extinction length $s_0 = Q$ is taken from equation (2.63). We then proceed along the lines of section 3.1, integrating both sides of the coupled Schwarzschild–Milne equations over the variables $0 < \tau < \infty$ and $-1 < \mu < 1$. The integrals over τ' which are independent of the transmission $T(\mu)$ can be performed explicitly, whereas the Q-dependence of the integrals involving $T(\mu)$ can be neglected in the scaling regime. We thus obtain the following equations for the functions $G^{(m)}(Q, \mu)$

$$\int_{-1}^{1} \frac{\mathrm{d}\mu}{2} G^{(m)}(Q, \mu) = Q\delta_{m,0} - Q \int_{-1}^{1} \frac{\mathrm{d}\mu}{2} \int_{0}^{1} \frac{\mathrm{d}\mu'}{2} \mu' T(\mu') p_m(\mu, \mu') G^{(m)}(Q, \mu')$$
$$+ \sum_{-\infty<k<+\infty} \int_{-1}^{1} \frac{\mathrm{d}\mu}{2} \int_{-1}^{1} \frac{\mathrm{d}\mu'}{2} p_m(\mu, \mu') \Lambda^{(m,k)}(Q, \mu') G^{(k)}(Q, \mu') \tag{3.10}$$

with

$$\Lambda^{(m,k)}(Q, \mu) = \frac{1}{\sqrt{1 + Q^2(1-\mu^2)}} \left(\frac{-Q\sqrt{1-\mu^2}}{1 + \sqrt{1 + Q^2(1-\mu^2)}} \right)^{|m-k|}$$
$$\times \begin{cases} (-1)^{m-k} & (m \leqslant k) \\ 1 & (m \geqslant k). \end{cases} \tag{3.11}$$

The solution of equation (3.10) in the scaling regime is as follows. Along the lines of section 2.7, we only keep the sectors $m = 0$ and $m = 1$, and only the leading amplitude in the expansion (2.58) in each sector, namely we set $G^{(0)}(Q, \mu) \approx c_{0,0}$, $G^{(1)}(Q, \mu) \approx c_{1,1}\sqrt{1-\mu^2}$. Inserting these expressions into (3.10), all integrals can be performed, to leading order in Q. We thus obtain $c_{1,1} \approx -(Q/2)(\tau^* - 1)c_{0,0}$, in agreement with equation (2.60), while equation (2.57) yields $\gamma_C(Q) \approx c_{0,0}$. After some algebra, we are left with the following scaling result:

$$\gamma_C(Q) \approx \frac{1}{\mathcal{T}/4 + Q\tau^*/3} \qquad (Q \ll 1, \mathcal{T} \ll 1). \tag{3.12}$$

The small-Q expansion of this prediction reads

$$\gamma_C(Q) \approx \frac{4}{\mathcal{T}} - \frac{16Q\tau^*}{3\mathcal{T}^2} + \cdots. \tag{3.13}$$

The width of the cone therefore scales as

$$\Delta Q \approx \frac{3\mathcal{T}}{4\tau^*} \tag{3.14}$$

in agreement with the general formula (2.67), together with the results (3.9).

The comment made at the end of section 3.1 still applies here. The scaling form of the enhanced backscattering cone in the regime of a large index mismatch does not depend at all on the anisotropy of the scattering cross section, apart from the simple power of τ^* already predicted by the diffusion approximation [31, 32].

4. Very anisotropic scattering

In this section we investigate the regime where the scattering cross section is very anisotropic, i.e. strongly peaked in a narrow cone of width $\Theta_{\text{rms}} \ll 1$ around the forward direction, with $\Theta_{\rm rms}^2 = \langle\Theta^2\rangle$. We thus have $1 - \varpi_1 \approx \Theta_{\rm rms}^2/2 \ll 1$, so that the anisotropy parameter $\tau^* \approx 2/\Theta_{\rm rms}^2$ is very large. The wavelength λ_0 of radiation in the medium, the scattering mean free path ℓ, and the transport mean free path ℓ^* can therefore be considered as three independent length scales ($\lambda_0 \ll \ell \ll \ell^*$), besides other characteristic lengths, such as the sample thickness L, and possibly the absorption length $L_{\rm abs}$.

The interest in this very anisotropic regime is twofold. First, we shall show that the radiative transfer problem in the absence of internal reflections is exactly solvable in this regime, just as it is for isotropic scattering, whereas the intermediate situation of a general anisotropy can only be treated numerically. Second, the dependence of physical quantities on anisotropy can be expected to yield the largest effects in the very anisotropic regime. We shall come back to this point in section 5.

4.1. The example of large spheres

We first recall how very anisotropic scattering can be realized experimentally. We consider the multiple scattering of light by large dielectric spheres, with radius a much larger than the wavelength λ_0 in the medium, i.e. with scale parameter $k_0a \gg 1$, so that geometrical optics can be used. Furthermore we assume that the optical index n_S of the spheres is very close to the mean index n_0 of the medium, i.e.

$$\frac{n_S}{n_0} = m_S = 1 + \delta_S \qquad (|\delta_S| \ll 1). \tag{4.1}$$

The study of the scattering cross section of electromagnetic waves by dielectric spheres is an old classical subject. The full solution was first derived by Mie in 1908. Reference [33] provides an extensive overview of this field. The regime $k_0a \gg 1$ and $|\delta_S| \ll 1$ still has to be split into several subcases, according to the value of the combination $|\delta_S|k_0a$. This can be understood as follows. In the framework of geometrical optics we can distinguish between the diffracted light, which is outgoing within an angle $\Theta_{\rm diffr} \sim 1/(k_0a)$, independent of δ_S, and the refracted light, which is outgoing within an angle $\Theta_{\rm refr} \sim |\delta_S|$, independent of k_0a.

From now on we concentrate our attention on the regime $|\delta_S| \ll 1$, $k_0a \gg 1$, and $|\delta_S|k_0a \gg 1$. In this regime we have $\Theta_{\rm diffr} \ll \Theta_{\rm refr} \ll 1$. The cross sections associated with each of the above processes asymptotically reads $\sigma_{\rm diffr} \approx \sigma_{\rm refr} \approx \pi a^2$, so that the total elastic cross section $\sigma \approx 2\pi a^2$ is twice the geometrical one. This is the *extinction paradox*. We make the following approximations. We neglect the diffraction phenomenon by setting $\Theta_{\rm diffr} = 0$. We treat the refracted light according to the laws of geometrical optics, as illustrated in figure 1. We neglect all the rays which are reflected at least once at the surface of the sphere, so that we only have to consider the refracted ray drawn on the figure. The corresponding phase function reads

$$p_{\rm refr}(\Theta) = \frac{4\sin\beta\cos\beta}{\sin\Theta}\left|\frac{{\rm d}\beta}{{\rm d}\Theta}\right| \tag{4.2}$$

with the notations of figure 1, which also imply

$$\Theta \approx -2\delta_S\tan\beta. \tag{4.3}$$

Equations (4.2), (4.3) lead to the Lorentzian-squared scaling form [33]

$$p_{\rm refr}(\Theta) \approx \frac{16\delta_S^2}{(\Theta^2 + 4\delta_S^2)^2}. \tag{4.4}$$

The cross section is thus strongly peaked in the forward direction, as anticipated.

We now turn to the evaluation of the coefficient $\varpi_{1,\rm refr}$ corresponding to refracted light. The integral $\langle\Theta^2\rangle_{\rm refr} = \int_{-\infty}^{\infty} \Theta^2 p_{\rm refr}(\Theta)\Theta\,{\rm d}\Theta$, with the phase function of equation (4.4),

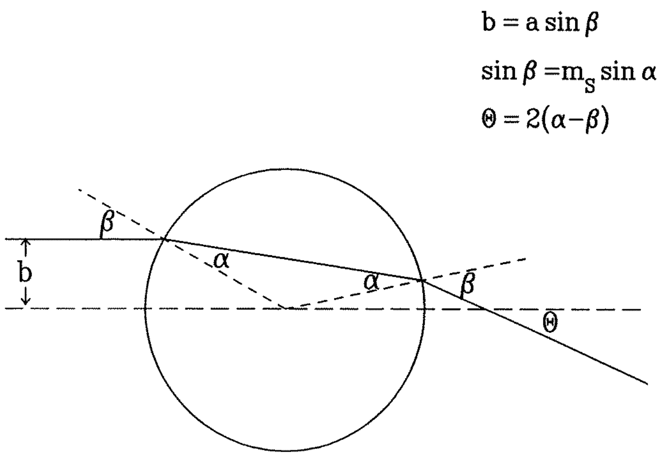

Figure 1. Laws of geometrical optics for the refraction of light by a large dielectric sphere.

is logarithmically divergent. A more careful treatment is therefore needed, which consists in using directly equation (4.2), choosing $u = \sin^2\beta$ as the integration variable. We thus obtain

$$\varpi_{1,\mathrm{refr}} = \frac{1}{3m_S^2} + \frac{4}{m_S^2}\int_0^{u_{\max}} u\,\mathrm{d}u\,\sqrt{(1-u)(m_S^2-u)} \tag{4.5}$$

with $u_{\max}$ being the smaller of both numbers 1 and m_S^2. The integral in (4.5) can be performed explicitly for both $m_S < 1$ and $m_S > 1$. In both cases we obtain

$$\varpi_{1,\mathrm{refr}} \approx 1 - 2\delta_S^2 \ln\frac{\Lambda}{|\delta_S|} \qquad \text{with } \Lambda = 2\mathrm{e}^{-3/2} \qquad (|\delta_S| \ll 1). \tag{4.6}$$

The anisotropy parameter can be derived from the above estimate, not forgetting that $\sigma_{\mathrm{diffr}} \approx \sigma_{\mathrm{refr}} \approx \sigma/2$ implies $1 - \varpi_1 = (1 - \varpi_{1,\mathrm{refr}})/2$. To leading order for $|\delta_S| \ll 1$, this yields

$$\tau^* = \frac{\ell^*}{\ell} \approx \frac{1}{\delta_S^2 \ln\frac{\Lambda}{|\delta_S|}}. \tag{4.7}$$

It is therefore evident that large spheres with a weak dielectric contrast provide a physical instance of very anisotropic scattering, as described at the beginning of this section, with the refraction angle $\Theta_{\mathrm{refr}} \sim |\delta_S| \ll 1$ playing the role of Θ_{rms}, up to a logarithmic correction. We shall come back to this last point in section 5.

4.2. Scaling limit of the Schwarzschild–Milne equation

We now show that the general formalism of section 2 undergoes important simplifications in the regime of very anisotropic scattering. These will allow us to solve the Schwarzschild–Milne equation in the absence of index mismatch at the interface. This analytical solution, to be described in sections 4.3 and 4.4, therefore shows that both limiting cases of isotropic scattering and of very anisotropic scattering are nearly on the same footing as far as the existence of exact results is concerned.

The simplifications in the regime of very anisotropic scattering can be understood in physical terms as follows. Consider the random sequence of scattering events experienced by a light ray. At every scattering event, the direction $\boldsymbol{n}$ of the ray is only modified by a slight amount of order Θ_{rms}, with $\Theta_{\mathrm{rms}}^2 \approx 2/\tau^*$. The extremity of the vector $\boldsymbol{n}$ therefore performs a Brownian motion on the unit sphere, with a small angular diffusivity $\varepsilon \sim \Theta_{\mathrm{rms}}^2$. A similar picture has long been used in the context of semiflexible polymer chains: the so-called Kratky–Porod theory models polymers as continuous persistent walks, whose unit tangent vectors obey a diffusion equation on the sphere (see [34] for a review).

In more quantitative terms, the relationship (2.7) between the $\boldsymbol{n}$-dependence of the specific intensity $I(\boldsymbol{r}, \boldsymbol{n})$ and the source function $\Gamma(\boldsymbol{r}, \boldsymbol{n})$ takes the form

$$\Gamma(\boldsymbol{r}, \boldsymbol{n}) \approx (1 + \varepsilon\Delta) I(\boldsymbol{r}, \boldsymbol{n}) \tag{4.8}$$

where Δ is the Legendre operator (Laplace operator on the unit sphere)

$$\Delta = \cot\theta \frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2}. \tag{4.9}$$

As a consequence, the integral operators $\mathcal{D}^{(m)}$, introduced in equation (2.15), simplify to the following differential operators

$$\mathcal{D}^{(m)} \approx 1 + \varepsilon\Delta^{(m)} \tag{4.10}$$

where

$$\Delta^{(m)} = \cot\theta \frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} - \frac{m^2}{\sin^2\theta} = (1 - \mu^2)\frac{\partial^2}{\partial\mu^2} - 2\mu\frac{\partial}{\partial\mu} - \frac{m^2}{1 - \mu^2} \tag{4.11}$$

is the Legendre operator in the sector defined by the azimuthal integer m.

The angular diffusivity ε is then determined by observing that $P_1(\mu) = \mu$ is an eigenfunction of $\Delta^{(0)}$, with eigenvalue (-2), and of $\mathcal{D}^{(0)}$, with eigenvalue ϖ_1, by virtue of equation (2.16). We thus identify $\varpi_1 = 1 - 2\varepsilon$, hence, using (2.20),

$$\varepsilon = \frac{1}{2\tau^*} = \frac{\ell}{2\ell^*} \approx \frac{\Theta_{\mathrm{rms}}^2}{4} \tag{4.12}$$

in agreement with the above heuristic estimate.

The radiative transfer equations (2.12) thus assume the differential form

$$2\tau^*\mu \frac{\mathrm{d}}{\mathrm{d}\tau} I^{(m)}(\tau, \mu) = \Delta^{(m)} I^{(m)}(\tau, \mu) \tag{4.13}$$

which demonstrates that quantities vary over length scales of order $\tau \sim \tau^*$, i.e. $z \sim \ell^*$. In the next two sections we present our exact analytical predictions concerning observables pertaining to optically thick slabs, in the regime of very anisotropic scattering, in the absence of internal reflections. The analysis roughly follows the presentation of section III of [15], devoted to the case of isotropic scattering.

4.3. Exact treatment in the absence of internal reflections: the homogeneous case

This section is devoted to an analytic determination of the solution $\Gamma_H(\tau, \mu)$ of the homogeneous Schwarzschild–Milne equation in the regime of very anisotropic scattering, and of the associated quantities of interest, τ_0 and $\tau_1(\mu)$.

It is convenient to consider the Laplace transform $g_H(s, \mu)$ of $\Gamma_H(\tau, \mu)$, defined as follows

$$g_H(s, \mu) = \int_0^\infty \mathrm{d}\tau\, \Gamma_H(\tau, -\mu)\mathrm{e}^{s\tau} \qquad (\mathrm{Re}\, s < 0). \tag{4.14}$$

The Schwarzschild–Milne equation (2.26), (2.28) can be recast as

$$g_H(s,\mu) = \begin{cases} \mathcal{D}^{(0)}\left[\dfrac{g_H(s,\mu)}{1+s\mu}\right] & (-1<\mu<0) \\ \mathcal{D}^{(0)}\left[\dfrac{g_H(s,\mu)-\tau^*\tau_1(\mu)/3}{1+s\mu}\right] & (0<\mu<1) \end{cases} \tag{4.15}$$

since we have $g_H(-1/\mu,\mu) = \tau^*\tau_1(\mu)/3$ for $0 < \mu < 1$, as a consequence of equations (2.35), (2.39).

We now expand (4.15), using the expression (4.10), (4.12) of the operator $\mathcal{D}^{(0)}$. We introduce the rescaled Laplace variable

$$\sigma = 2s\tau^* \tag{4.16}$$

as well as a new unknown function $h_H(\sigma,\mu)$, defined as follows

$$g_H(s,\mu) = \begin{cases} 4\tau^{*2}(1+s\mu)h_H(\sigma,\mu) & (-1<\mu<0) \\ 4\tau^{*2}(1+s\mu)h_H(\sigma,\mu)+\tau^*\tau_1(\mu)/3 & (0<\mu<1). \end{cases} \tag{4.17}$$

We assume that $\tau_1(\mu)$ vanishes faster than linearly in μ for $\mu \to 0$. This hypothesis will be checked *a posteriori*, as it will turn out that $\tau_1(\mu) \sim \mu^{3/2}$ for $\mu \to 0$ (see equation (4.55)). The function $h_H(\sigma,\mu)$ is then continuously differentiable as a function of μ, and it obeys the following equation

$$(\Delta^{(0)} - \sigma\mu)h_H(\sigma,\mu) = \begin{cases} 0 & (-1<\mu<0) \\ \tau_1(\mu)/6 & (0<\mu<1). \end{cases} \tag{4.18}$$

By means of the change of variable

$$\mu = \tanh x \tag{4.19}$$

equation (4.18) can be recast in the form of the following inhomogeneous Schrödinger equation on the real x-line

$$h_H''(\sigma,x) - \sigma V(x)h_H(\sigma,x) = \begin{cases} 0 & (x<0) \\ V(x)\nu(x)/6 & (x\geqslant 0). \end{cases} \tag{4.20}$$

In this formula the primes denote differentiation with respect to x. The potential

$$V(x) = \tanh x(1-\tanh^2 x) \tag{4.21}$$

is an odd function of x, such that $V(x)\,\mathrm{d}x = \mu\,\mathrm{d}\mu$, and we have set

$$\tau_1(\mu) = \mu\nu(x) \qquad (0<\mu<1, x>0). \tag{4.22}$$

Equation (4.20) is a self-consistent equation for the two unknown functions $h_H(\sigma,x)$ and $\nu(x)$. Analyticity properties in the complex σ-variable turn out to allow for an exact analytical solution of this equation.

We introduce a basis of two elementary solutions $\{u_1(\sigma,x), u_2(\sigma,x)\}$ of the (homogeneous) Schrödinger equation

$$u''(\sigma,x) - \sigma V(x)u(\sigma,x) = 0 \tag{4.23}$$

with the following asymptotic behaviour

$$u_1(\sigma,x) \approx 1 \qquad u_2(\sigma,x) \approx x \qquad (x\to-\infty) \tag{4.24}$$

up to exponentially small corrections. Similarly we introduce the basis $\{v_1(\sigma,x), v_2(\sigma,x)\}$, with the asymptotic behaviour

$$v_1(\sigma,x) \approx 1 \qquad v_2(\sigma,x) \approx x \qquad (x\to\infty). \tag{4.25}$$

The Schrödinger equation (4.23) admits solutions with the above boundary conditions, since the potential $V(x)$ vanishes exponentially as $x \to \pm\infty$. The four solutions above are entire functions of σ, i.e. they are analytic in the whole σ-plane. Furthermore they are related by the following identities

$$v_1(\sigma, x) = u_1(-\sigma, -x) \qquad v_2(\sigma, x) = -u_2(-\sigma, -x) \tag{4.26}$$

because the potential $V(x)$ is odd.

We recall that, if $u(x)$ and $v(x)$ are any two solutions of the Schrödinger equation (4.23), with the same σ, their *Wronskian*

$$W\{u, v\} = u(x)v'(x) - u'(x)v(x) \tag{4.27}$$

is independent of x. The boundary conditions (4.24), (4.25) imply that both bases of functions have unit Wronskian, namely

$$W\{u_1, u_2\} = W\{v_1, v_2\} = 1. \tag{4.28}$$

For generic values of the parameter σ, both bases of solutions are related by a 2×2 *transfer matrix* of the form

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} F(\sigma) & G(\sigma) \\ H(\sigma) & F(-\sigma) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{4.29}$$

The three functions which enter equation (4.29) are entire functions of σ; the determinant of the matrix is $F(\sigma)F(-\sigma) - G(\sigma)H(\sigma) = 1$; as a consequence of equation (4.26), $G(\sigma)$ and $H(\sigma)$ are even functions of σ.

The above functions also govern the non-trivial asymptotic behaviour of the bases of solutions

$$\begin{aligned} &u_1(\sigma, x) \approx F(-\sigma) - G(\sigma)x \qquad u_2(\sigma, x) \approx -H(\sigma) + F(\sigma)x \qquad (x \to \infty) \\ &v_1(\sigma, x) \approx F(\sigma) + G(\sigma)x \qquad v_2(\sigma, x) \approx H(\sigma) + F(-\sigma)x \qquad (x \to -\infty) \end{aligned} \tag{4.30}$$

as well as their mixed Wronskians

$$\begin{aligned} &W\{u_1, v_1\} = G(\sigma) \qquad W\{u_1, v_2\} = F(-\sigma) \\ &W\{u_2, v_1\} = -F(\sigma) \qquad W\{u_2, v_2\} = -H(\sigma). \end{aligned} \tag{4.31}$$

The first terms of the Taylor expansion of $u_1(\sigma, x)$ and $u_2(\sigma, x)$ around $\sigma = 0$ read

$$\begin{aligned} &u_1(\sigma, x) = 1 - \frac{\sigma}{2}(1 + \tanh x) + \cdots \\ &u_2(\sigma, x) = x + \sigma\left(\frac{x}{2}(1 - \tanh x) + \ln(2\cosh x)\right) + \cdots. \end{aligned} \tag{4.32}$$

We have also determined the terms of order σ^2, which are too lengthy expressions to be reported here. They imply

$$\begin{aligned} &F(\sigma) = 1 + \sigma + \sigma^2/2 + \cdots \qquad G(\sigma) = \sigma^2/3 + \cdots \\ &H(\sigma) = (7/6 - \pi^2/36)\sigma^2 + \cdots. \end{aligned} \tag{4.33}$$

On the other hand, large values of the complex parameter σ correspond to the *semiclassical* regime for the Schrödinger equation (4.23), where the behaviour of the functions $u_1(\sigma, x)$ and $u_2(\sigma, x)$ can be derived by means of a WKB-like approximation. This regime is analysed in detail in appendix B. Let us mention that the wavefunctions display oscillations when either $\sigma > 0$ and $x < 0$ or $\sigma < 0$ and $x > 0$, whereas they are growing or decaying exponentially in the other two cases.

The function $G(\sigma)$ deserves some more attention, since it will play a central role in the following. $G(\sigma)$ can be viewed as the *functional determinant* of the Schrödinger equation (4.23), in the following sense. Assume σ is such that $G(\sigma) = 0$. We have then

$$v_1(\sigma, x) = F(\sigma)u_1(\sigma, x) \qquad u_1(\sigma, x) = F(-\sigma)v_1(\sigma, x) \qquad F(\sigma)F(-\sigma) = 1. \tag{4.34}$$

In other words, for such a σ, the Schrödinger equation (4.23) has a bounded solution over the whole real line. Throughout the following, using slightly improper terms, such a value of σ is called an *eigenvalue* of the Schrödinger equation, and the corresponding function $v_1(\sigma, x)$ is referred to as the associated *eigenfunction*. The semiclassical analysis of appendix B demonstrates that there is an infinite sequence of real eigenvalues. We label them by an integer $-\infty < n < \infty$, so that $\sigma_n > 0$ for $n \geqslant 1$, $\sigma_0 = 0$, and $\sigma_{-n} = -\sigma_n$. The associated eigenfunctions $v_1(\sigma_n, x)$ are orthogonal with respect to the following indefinite metric

$$\int_{-\infty}^{\infty} v_1(\sigma_m, x)v_1(\sigma_n, x)V(x)\,\mathrm{d}x = N_n\delta_{m,n} \qquad (-\infty < m, n < \infty). \tag{4.35}$$

The squared norms N_n obey the symmetry property

$$\frac{N_{-n}}{F(-\sigma_n)} = -\frac{N_n}{F(\sigma_n)} \tag{4.36}$$

as a consequence of (4.34). The eigenfunction $v_1(0, x) = 1$ associated with the zero mode $\sigma_0 = 0$ is peculiar, since its squared norm reads $N_0 = 0$. As a consequence, the basis of eigenfunctions $\{v_1(\sigma_n, x), n \neq 0\}$ only spans the set of bounded functions $f(x)$ on the real x-line such that

$$\int_{-\infty}^{\infty} f(x)V(x)\,\mathrm{d}x = 0. \tag{4.37}$$

For such functions, we have

$$f(x) = \sum_{n\neq 0} c_n v_1(\sigma_n, x) \tag{4.38}$$

with

$$c_n = \frac{1}{N_n}\int_{-\infty}^{\infty} v_1(\sigma_n, x)f(x)V(x)\,\mathrm{d}x. \tag{4.39}$$

The determination of the contribution of the zero mode to functions $f(x)$ which do not obey the condition (4.37), such as, for example, a constant, requires more care. An elegant way of dealing with this problem consists in introducing a deformation parameter κ, as we shall see in section 4.4.

It is advantageous to factor the entire function $G(\sigma)$ as follows

$$G(\sigma) = \frac{\sigma^2}{3}P(\sigma)P(-\sigma) \tag{4.40}$$

with the notation

$$P(\sigma) = \prod_{n\geqslant 1}\left(1 + \frac{\sigma}{\sigma_n}\right). \tag{4.41}$$

The explicit solution of the inhomogeneous Schrödinger equation (4.20) now goes as follows. Since $h_H(\sigma, x)$ is a regular function of x in the $x \to -\infty$ limit, it is proportional to $u_1(\sigma, x)$ for $x < 0$, namely

$$h_H(\sigma, x) = a_H(\sigma)u_1(\sigma, x) \qquad (x < 0). \tag{4.42}$$

For $x > 0$ we solve equation (4.20) by *varying the constants*, namely we look for a solution of the form

$$h_H(\sigma, x) = b_H(\sigma, x)v_1(\sigma, x) + c_H(\sigma, x)v_2(\sigma, x) \qquad (x > 0). \tag{4.43}$$

The unknown *constants* $b_H(\sigma, x)$ and $c_H(\sigma, x)$ obey the requirements

$$b'_H(\sigma, x)v_1(\sigma, x) + c'_H(\sigma, x)v_2(\sigma, x) = 0 \tag{4.44a}$$

$$b'_H(\sigma, x)v'_1(\sigma, x) + c'_H(\sigma, x)v'_2(\sigma, x) = V(x)\nu(x)/6 \tag{4.44b}$$

where the primes again denote differentiations with respect to x. equation (4.44*a*) is a constraint imposed *a priori*, in order to break the large redundance of the representation (4.43); equation (4.44*b*) is then a consequence of equation (4.20). Equation (4.44) can be solved for b'_H and c'_H, using equation (4.28). Integrating the expressions thus obtained, we get

$$\begin{aligned} b_H(\sigma, x) &= b_H(\sigma, \infty) + \int_x^\infty v_2(\sigma, y)\nu(y)V(y)\,\mathrm{d}y/6 \\ c_H(\sigma, x) &= -\int_x^\infty v_1(\sigma, y)\nu(y)V(y)\,\mathrm{d}y/6. \end{aligned} \tag{4.45}$$

Indeed there cannot be a non-zero constant $c_H(\sigma, \infty)$, because this would correspond to an unacceptable singular solution of the form $h_H(\sigma, x) \sim x \sim \ln(1-\mu)$ for $\mu \to 1$.

Both expressions (4.42) and (4.43) have to match at $x = 0$, together with their first derivatives. These two conditions determine $b_H(\sigma, 0)$ and $c_H(\sigma, 0)$, and some algebra then leads to the identity

$$G(\sigma)a_H(\sigma) = -c_H(\sigma, 0) = \int_0^\infty v_1(\sigma, x)\nu(x)V(x)\,\mathrm{d}x/6. \tag{4.46}$$

We can argue on equation (4.46) as follows. Since $g_H(s, \mu)$ is analytic in the half-plane $\mathrm{Re}\, s < 0$, $a_H(\sigma)$ has the same property for $\mathrm{Re}\,\sigma < 0$, so that the zeros $-\sigma_n$ of $G(\sigma)$ cannot be poles of $a_H(\sigma)$. They are therefore zeros of $c_H(\sigma, 0)$. On the other hand, the σ_n's are not zeros of $c_H(\sigma, 0)$, since the integral expression in (4.46) is positive for $\sigma > 0$. Hence they are poles of $a_H(\sigma)$. The final step concerns the small-σ behaviour of the quantities involved in (4.46). The asymptotic behaviour (2.29) of $\Gamma_H(\tau, \mu)$ yields the following double-pole structure for its Laplace transform near $s = 0$

$$g_H(s, \mu) = \frac{1}{s^2} - \frac{\tau^*(\tau_0 + \mu)}{s} + \mathcal{O}(1) \qquad (s \to 0) \tag{4.47}$$

which implies the following small-σ behaviour of $a_H(\sigma)$

$$a_H(\sigma) = \frac{1}{\sigma^2} + \frac{1-\tau_0}{2\sigma} + \mathcal{O}(1) \qquad (\sigma \to 0). \tag{4.48}$$

We thus obtain finally

$$a_H(\sigma) = \frac{1}{\sigma^2 P(-\sigma)} \qquad -c_H(\sigma, 0) = \frac{P(\sigma)}{3}. \tag{4.49}$$

Equation (4.49) can be considered as an explicit result. Indeed $P(\sigma)$, defined in (4.41), is for all purposes a known function, since the eigenvalues σ_n can be determined numerically, essentially with arbitrary accuracy, via the partial-wave expansion procedure of appendix A. Furthermore the semiclassical analysis of appendix B determines the asymptotic behaviour of the eigenvalues in the regime of large quantum numbers ($n \gg 1$).

Table 2. Comparison of the numerical values of various quantities of interest from the exact solutions in the absence of internal reflections. First row: isotropic scattering, after [15]; second row: very anisotropic scattering (this work). $\tau_0\ell^*$ is the thickness of a skin layer; $\tau_1(1)$ and $\gamma(1,1)$ respectively yield the transmitted and reflected intensities in the normal direction; $B(0)$ is the peak value of the enhancement factor at the top of the backscattering cone; $\tau^*\Delta Q = k_1\ell^*\Delta\theta$ is the dimensionless width of this backscattering cone. The third row gives the relative difference of the second case with respect to the first one.

	τ_0	$\tau_1(1)$	$\gamma(1,1)$	$B(0)$	$\tau^*\Delta Q$
Isotropic ($\tau^* = 1$)	0.710 446	5.036 475	4.227 681	1.881 732	1/2
Very anisotropic ($\tau^* \gg 1$)	0.718 211	5.138 580	4.889 703	2	0.555 543
Δ (%)	1.1	2.0	15.7	6.3	11.1

As a first consequence of the above results, we can determine the reduced thickness τ_0 of a skin layer, i.e. the reduced extrapolation length, by comparing the small-σ behaviour of the expression (4.49) for $a_H(\sigma)$ with the expansion (4.48). We thus obtain

$$\tau_0 = 1 - 2\sum_{n\geqslant 1}\frac{1}{\sigma_n} = 0.718\,211\,64. \tag{4.50}$$

The series converges, since σ_n grows as n^2, according to the semiclassical estimate (B.21). The number given in equation (4.50), as well as all the subsequent ones, has been obtained by means of the partial-wave expansion described in appendix A. This number gives an idea of the accuracy of this approach. The most significant numerical results are listed in table 2, together with their counterparts in the case of isotropic scattering.

Second, the determination of $\tau_1(\mu)$ goes as follows. We recall that this quantity is related to $\nu(x)$ by equation (4.22). Equations (4.46), (4.49) yield

$$\int_0^\infty v_1(\sigma, x)\nu(x)V(x)\,\mathrm{d}x = 2P(\sigma). \tag{4.51}$$

The small-σ expansion (4.32) of $u_1(\sigma, x)$, together with (4.26), allows us to recover the sum rules (2.38), (2.42) in the following form

$$\int_0^\infty \nu(x)V(x)\,\mathrm{d}x = 2 \qquad \int_0^\infty \nu(x)\tanh x\, V(x)\,\mathrm{d}x = 2\tau_0. \tag{4.52}$$

On the other hand, the semiclassical estimates (B.18), (B.20) of $v_1(\sigma, x)$ and $P(\sigma)$ for large values of $\sigma = K^2$ imply

$$\int_0^\infty \nu(x)\,\mathrm{Ai}(K^{2/3}x)x\,\mathrm{d}x \approx (6/\pi)^{1/2}K^{-5/3} \qquad (K \gg 1). \tag{4.53}$$

This estimate yields, by means of (B.25) for $s = \frac{3}{2}$, the small-x behaviour of $\nu(x)$, i.e.

$$\nu(x) \approx 6(2x/\pi)^{1/2} \qquad (x \ll 1). \tag{4.54}$$

We thus obtain the following universal scaling behaviour of the τ_1-function in the case of very anisotropic scattering

$$\tau_1(\mu) \approx 6(2/\pi)^{1/2}\mu^{3/2} \qquad (\mu \ll 1). \tag{4.55}$$

This novel result is in contrast with the linear behaviour $\tau_1(\mu) \approx \mu\sqrt{3}$ observed for isotropic scattering. The law (4.55) also confirms the hypothesis made in the beginning of this section, namely that $\tau_1(\mu)$ vanishes faster than linearly in μ.

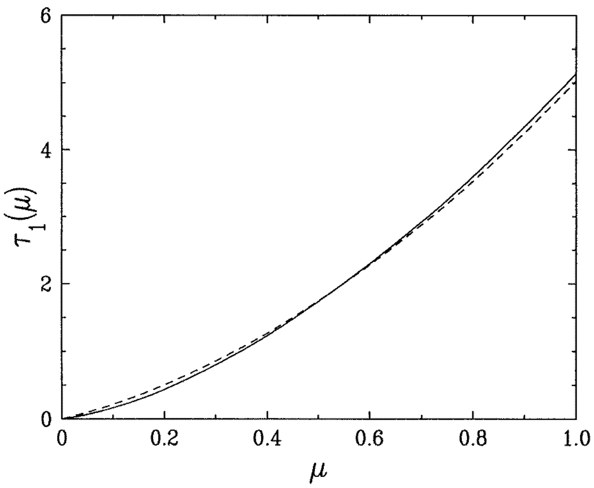

Figure 2. Plot of exact expressions for $\tau_1(\mu)$ in the absence of internal reflections. Broken curve: isotropic scattering, after [15]. Full curve: very anisotropic scattering (this work).

Finally, we can extract the full functions $\nu(x)$ and $\tau_1(\mu)$ from (4.51) by means of the inversion formula (4.38), (4.39). We obtain

$$\nu(x) = 3(\tau_0 + \tanh x) + 2\sum_{n\geqslant 1} \frac{P(\sigma_n)}{N_n} v_1(\sigma_n, x) \tag{4.56}$$

i.e.

$$\frac{\tau_1(\mu)}{\mu} = 3(\tau_0 + \mu) + 2\sum_{n\geqslant 1} \frac{P(\sigma_n)}{N_n} v_1(\sigma_n, \operatorname{arg\,tanh}\mu). \tag{4.57}$$

The semiclassical analysis of appendix B implies that the contribution of the nth mode to the results (4.56), (4.57) falls off exponentially with n for $x > 0$, according to $\exp[-K_n(I - I(x))]$. This exponential convergence disappears as $x \to 0$, where equations (4.54), (4.55) apply. The explicit terms in front of the sums in equations (4.56), (4.57), corresponding to the contribution of the zero mode ($n = 0$), have been anticipated from results to be derived in section 4.4.

Figure 2 shows a plot of the function $\tau_1(\mu)$, for both isotropic scattering [15] and very anisotropic scattering (equation (4.57)). The maximal values $\tau_1(1)$ for both cases are given in table 2. The difference between both limiting cases is remarkably small.

4.4. Exact treatment in the absence of internal reflections: the inhomogeneous case

In this section we derive analytical expressions in the regime of very anisotropic scattering of the special solution $\Gamma_S(\tau, \mu, \mu_a)$, with its by-product the bistatic coefficient $\gamma(\mu_a, \mu_b)$.

In analogy with equation (4.14), we define the Laplace transform of the source function as follows:

$$g_S(s, \mu, \mu_a) = \int_0^\infty \mathrm{d}\tau\, \Gamma_S(\tau, -\mu, \mu_a)\mathrm{e}^{s\tau} \qquad (\operatorname{Re} s < 0). \tag{4.58}$$

The Schwarzschild–Milne equation (2.26), (2.28) can be recast as

$$g_S(s, \mu, \mu_a) = \begin{cases} \dfrac{\mu_a p_0(-\mu, \mu_a)}{1 - s\mu_a} + \mathcal{D}^{(0)}\left[\dfrac{g_S(s, \mu, \mu_a)}{1 + s\mu}\right] & (-1 < \mu < 0) \\ \mathcal{D}^{(0)}\left[\dfrac{g_S(s, \mu, \mu_a) - \gamma(\mu, \mu_a)}{1 + s\mu}\right] & (0 < \mu < 1) \end{cases} \tag{4.59}$$

since we have $g_S(-1/\mu, \mu, \mu_a) = \gamma(\mu_a)$ for $0 < \mu < 1$, as a consequence of equation (2.33).

In analogy with section 4.3, we expand equation (4.59), using equations (4.10), (4.12). We use the rescaled variable σ of equation (4.16), and we introduce a new unknown function $h_S(\sigma, \mu, \mu_a)$, defined as follows

$$g_S(s, \mu, \mu_a) = \begin{cases} 2\tau^*(1 + s\mu)h_S(\sigma, \mu, \mu_a) & (-1 < \mu < 0) \\ 2\tau^*(1 + s\mu)h_S(\sigma, \mu, \mu_a) + \gamma(\mu, \mu_a) & (0 < \mu < 1). \end{cases} \tag{4.60}$$

We assume that $\gamma(\mu_a, \mu_b)$ vanishes faster than linearly as μ_a or $\mu_b \to 0$. Using the change of variable (4.19), we are again left with an inhomogeneous Schrödinger equation, namely

$$h_S''(\sigma, x, x_a) - \sigma V(x)h_S(\sigma, x, x_a) = \begin{cases} -2\mu_a\delta(x + x_a) & (x < 0) \\ \mu_a V(x)\rho(x, x_a) & (x > 0) \end{cases} \tag{4.61}$$

where we have set

$$\gamma(\mu, \mu_a) = \mu\mu_a\rho(x, x_a) \qquad (\mu = \tanh x > 0, \mu_a = \tanh x_a > 0). \tag{4.62}$$

Now, in order to deal with the problem of the zero modes of the Schrödinger equation (4.23), we introduce a continuous *deformation parameter* $\kappa > 0$ as follows. We consider the deformed Schrödinger equation

$$u''(\sigma, x) - (\sigma V(x) + \kappa^2 W(x))u(\sigma, x) = 0 \tag{4.63}$$

with

$$W(x) = (1 - \tanh^2 x)^2. \tag{4.64}$$

The eigenvalues with label $n \neq 0$ acquire a regular κ-dependence of the form

$$\sigma_n(\kappa) = -\sigma_{-n}(\kappa) = \sigma_n + \mathcal{O}(\kappa^2) \tag{4.65}$$

as well as the associated eigenfunctions $v_1(\kappa, \sigma_n(\kappa), x)$, whereas the double degeneracy of the zero mode $\sigma_0 = 0$ is lifted into the following two exact eigenvalues and eigenfunctions of (4.63):

$$\sigma_{\pm 0}(\kappa) = \pm 2\kappa \qquad v_1(\kappa, \sigma_{\pm 0}(\kappa), x) = \exp(\pm\kappa(1 - \tanh x)) = \exp(\pm\kappa(1 - \mu)) \tag{4.66}$$

with squared norms

$$N_{\pm 0}(\kappa) = \pm\frac{e^{\pm 2\kappa}}{\kappa}\left(\frac{\sinh 2\kappa}{2\kappa} - \cosh 2\kappa\right). \tag{4.67}$$

The introduction of labels ± 0 is consistent with setting $n = 0$ in formulae such as (4.36) or (4.65), rather than with the standard arithmetics of integers!

For any non-zero κ, the set of eigenfunctions $\{v_1(\kappa, \sigma_n(\kappa), x)\}$, where the label n runs over the non-zero algebraic integers ($n \neq 0$) plus both values $n = \pm 0$, now spans the whole space of bounded functions $f(x)$ on the real line. The difficulty of the constraint (4.37), due to the vanishing norm of the zero mode at $\kappa = 0$, is thus cured in a natural way.

The mixed Wronskian $G(\kappa, \sigma) = W\{u_1, v_1\}$ can still be factorized over its zeros, in analogy with equation (4.40), namely

$$G(\kappa, \sigma) = G(\kappa, 0)R(\kappa, \sigma)R(\kappa, -\sigma) \tag{4.68}$$

with

$$R(\kappa, \sigma) = \prod_{n \geqslant 0}\left(1 + \frac{\sigma}{\sigma_n(\kappa)}\right). \tag{4.69}$$

The prefactor $G(\kappa, 0)$ of (4.68) is a non-trivial function of κ. Indeed this quantity can be viewed as the functional determinant of the Schrödinger equation (4.63) with $\sigma = 0$, namely

$$u''(\sigma, x) - \kappa^2 W(x) u(\sigma, x) = 0. \tag{4.70}$$

Equation (4.70) is equivalent to the spheroidal equation studied by Meixner and Schäfke [35]. Some properties of this equation have been studied in detail [36], in an investigation of nematic phases of semiflexible polymer chains. The occurrence of (4.70) in that context is related to the Kratky–Porod description of persistent chains, mentioned in section 4.2.

Equation (4.70) has a discrete spectrum of imaginary eigenvalues of the form $\kappa = \pm \mathrm{i}\xi_n$ ($n \geqslant 1$), as shown by the semiclassical analysis of appendix B. On the other hand, the regularity of $G(\kappa, \sigma)$ at $\kappa = 0$ implies the small-κ behaviour $G(\kappa, 0) \approx 4\kappa^2/3$, hence

$$G(\kappa, 0) = \frac{4\kappa^2}{3} \prod_{n \geqslant 1} \left(1 + \frac{\kappa^2}{\xi_n^2}\right). \tag{4.71}$$

The solution of the κ-dependent deformed inhomogeneous Schrödinger equation (4.61) then follows the lines of section 4.3. The particular values $x = -x_a < 0$ and $x = 0$ define three sectors, in which we look for a solution of the following form

$$h_S(\kappa, \sigma, x, x_a) = \begin{cases} a_S(\kappa, \sigma, x_a) u_1(\kappa, \sigma, x) & (x < -x_a) \\ b_S(\kappa, \sigma, x_a) u_1(\kappa, \sigma, x) + c_S(\kappa, \sigma, x_a) u_2(\kappa, \sigma, x) & (-x_a < x < 0) \\ d_S(\kappa, \sigma, x, x_a) v_1(\kappa, \sigma, x) + e_S(\kappa, \sigma, x, x_a) v_2(\kappa, \sigma, x) & (x > 0). \end{cases} \tag{4.72}$$

The *constants* which enter the last of these expressions obey the conditions

$$\begin{aligned} &d'_S(\kappa, \sigma, x, x_a) v_1(\kappa, \sigma, x) + e'_S(\kappa, \sigma, x, x_a) v_2(\kappa, \sigma, x) = 0 \\ &d'_S(\kappa, \sigma, x, x_a) v'_1(\kappa, \sigma, x) + e'_S(\kappa, \sigma, x, x_a) v'_2(\kappa, \sigma, x) = \mu_a V(x) \rho(\kappa, x, x_a) \end{aligned} \tag{4.73}$$

hence

$$\begin{aligned} d_S(\kappa, \sigma, x, x_a) &= d_S(\kappa, \sigma, \infty, x_a) + \mu_a \int_x^\infty v_2(\kappa, \sigma, y) \rho(\kappa, y, x_a) V(y)\, \mathrm{d}y \\ e_S(\kappa, \sigma, x, x_a) &= -\mu_a \int_x^\infty v_1(\kappa, \sigma, y) \rho(\kappa, y, x_a) V(y)\, \mathrm{d}y. \end{aligned} \tag{4.74}$$

On the other hand, the matching of the solution (4.72) at $x = -x_a$ yields

$$\begin{aligned} b_S(\kappa, \sigma, x_a) &= a_S(\kappa, \sigma, x_a) + 2\mu_a u_2(\kappa, \sigma, x_a) \\ c_S(\kappa, \sigma, x_a) &= -2\mu_a u_1(\kappa, \sigma, x_a) \end{aligned} \tag{4.75}$$

whereas its matching at $x = 0$ leads to

$$\begin{aligned} d_S(\kappa, \sigma, 0, x_a) &= F(\kappa, -\sigma) b_S(\kappa, \sigma, x_a) - H(\kappa, \sigma) c_S(\kappa, \sigma, x_a) \\ -e_S(\kappa, \sigma, 0, x_a) &= G(\kappa, \sigma) b_S(\kappa, \sigma, x_a) - F(\kappa, \sigma) c_S(\kappa, \sigma, x_a) \end{aligned} \tag{4.76}$$

with the notations (4.29). We are thus left with

$$G(\kappa, \sigma) a_S(\kappa, \sigma, x_a) = -2\mu_a v_1(\kappa, \sigma, -x_a) - e_S(\kappa, \sigma, 0, x_a). \tag{4.77}$$

We can now follow the approach used on equation (4.46). Since $a_S(\kappa, \sigma, x_a)$ is holomorphic in the half-plane $\mathrm{Re}\,\sigma < 0$, the zeros $-\sigma_n(\kappa)$ for $n \geqslant 0$ of $G(\kappa, \sigma)$ cannot be poles of $a_S(\kappa, \sigma, x_a)$. Hence they are zeros of the right-hand side of (4.77).

We are therefore left with the problem of finding $e_S(\kappa, \sigma, 0, x_a)$, an entire function of σ, from the knowledge of its values on the sequence of points $\sigma = -\sigma_n(\kappa)$ $(n \geqslant 0)$, together with a natural assumption of minimal growth at infinity compatible with these data. This is a generalization to an entire function of the problem of finding a polynomial $Q(z)$ with minimal degree, knowing its values at N points, namely $Q(z_n) = Q_n$ for $1 \leqslant n \leqslant N$. It is useful to view the z_n's as the zeros of the normalized polynomial

$$P(z) = \prod_{1 \leqslant n \leqslant N} (z - z_n). \tag{4.78}$$

The solution $Q(z)$ with minimal degree (generically $N - 1$) is given by the following *Lagrange interpolation formula*:

$$Q(z) = \sum_{1 \leqslant n \leqslant N} Q_n \prod_{1 \leqslant m \neq n \leqslant N} \frac{z - z_m}{z_n - z_m} = P(z) \sum_{1 \leqslant n \leqslant N} \frac{Q_n}{(z - z_n) P'(z_n)}. \tag{4.79}$$

Extending (4.79) to the present case of an infinite sequence of data for an unknown entire function, we obtain

$$-e_S(\kappa, \sigma, 0, x_a) = 2\mu_a R(\kappa, \sigma) \sum_{n \geqslant 0} \frac{v_1(\kappa, -\sigma_n(\kappa), -x_a)}{(\sigma + \sigma_n(\kappa))(\mathrm{d}R/\mathrm{d}\sigma)(\kappa, -\sigma_n(\kappa))} \tag{4.80}$$

or equivalently, using a generalization of the identity (C.6) to $\kappa \neq 0$, together with the definition (4.68),

$$-e_S(\kappa, \sigma, 0, x_a) = -2\mu_a G(\kappa, 0) R(\kappa, \sigma) \sum_{n \geqslant 0} \frac{R(\kappa, \sigma_n(\kappa)) v_1(\kappa, \sigma_n(\kappa), x_a)}{(\sigma + \sigma_n(\kappa)) N_n(\kappa)}. \tag{4.81}$$

Finally, we can derive an explicit expression for $\rho(\kappa, x, x_a)$, by means of an inversion formula analogous to equations (4.38), (4.39), namely

$$\begin{aligned} \rho(\kappa, x, x_a) = -2G(\kappa, 0) \sum_{m,n \geqslant 0} \frac{R(\kappa, \sigma_m(\kappa)) R(\kappa, \sigma_n(\kappa))}{\sigma_m(\kappa) + \sigma_n(\kappa)} \frac{v_1(\kappa, \sigma_m(\kappa), x)}{N_m(\kappa)} \frac{v_1(\kappa, \sigma_n(\kappa), x_a)}{N_n(\kappa)} \\ - \sum_{n \geqslant 0} \frac{v_1(\kappa, \sigma_n(\kappa), x) v_1(\kappa, \sigma_n(\kappa), -x_a) + v_1(\kappa, \sigma_n(\kappa), x_a) v_1(\kappa, \sigma_n(\kappa), -x)}{N_n(\kappa)}. \end{aligned} \tag{4.82}$$

This central result is manifestly symmetric under the exchange of x and x_a, as it should be. This symmetry property is a valuable check of the whole approach, since both arguments x and x_a have played uneven roles throughout the derivation.

We are now able to take the physical $\kappa \to 0$ limit of the above results. In this regime equation (4.81) can be recast as

$$-e_S(\sigma, 0, x_a) = 2\mu_a P(\sigma) \left(1 - \frac{\sigma}{3} \sum_{n \geqslant 1} \frac{\sigma_n P(\sigma_n) v_1(\sigma_n, x_a)}{(\sigma + \sigma_n) N_n} \right). \tag{4.83}$$

First, we are now able to complete the proof of the anticipated result (4.56), (4.57), by inserting (4.83) into (4.77), expanding the latter equation for $\kappa \to 0$ to the first non-trivial order as $\sigma \to 0$, and comparing the result with the estimate

$$a_S(\sigma, x_a) \approx -\frac{\tau_1(\mu_a)}{\sigma} \qquad (\sigma \to 0). \tag{4.84}$$

Second, the small-σ expansion of (4.83) allows us to recover the sum rules (2.37), (2.41) in the following form:

$$\int_0^\infty \rho(x, x_a) V(x)\,\mathrm{d}x = 2 \qquad \int_0^\infty \rho(x, x_a) \tanh x\, V(x)\,\mathrm{d}x = 2\nu(x_a)/3 - 2\tanh x_a. \tag{4.85}$$

On the other hand, for large values of $\sigma = K^2$, we can use the semiclassical estimates (B.18), (B.20), (B.22), setting $\sigma_n = k^2$, in order to transform equations (4.74), (4.83) into

$$\int_0^\infty \rho(x, x_a)\,\mathrm{Ai}(K^{2/3}x)x\,\mathrm{d}x \approx (2/\pi)K^{1/3}\int_0^\infty \frac{k^{2/3}}{k^2+K^2}\,\mathrm{Ai}(k^{2/3}x_a)\,\mathrm{d}k. \tag{4.86}$$

This estimate shows that $xx_a\rho(x, x_a)$ is a homogeneous function of its arguments with degree zero, when both of them are small, i.e. $xx_a\rho(x, x_a) = g(x/x_a)$. The rescaling of (4.86) according to $z = K^{2/3}x = k^{2/3}x_a$ then yields, by means of a mere identification of both integrands, using (B.25), the expression $g(z) = (3/\pi)z^{3/2}/(z^3+1)$, implying the following scaling behaviour

$$\rho(x_a, x_b) \approx \frac{3(x_a x_b)^{1/2}}{\pi(x_a^3 + x_b^3)} \qquad (x_a, x_b \ll 1) \tag{4.87}$$

or equivalently

$$\gamma(\mu_a, \mu_b) \approx \frac{3(\mu_a\mu_b)^{3/2}}{\pi(\mu_a^3 + \mu_b^3)} \qquad (\mu_a, \mu_b \ll 1). \tag{4.88}$$

This novel result is in contrast with the rational behaviour $\gamma(\mu_a, \mu_b) \approx \mu_a\mu_b/(\mu_a + \mu_b)$ in the case of isotropic scattering. The law (4.88) confirms the hypothesis made at the beginning of this section, namely that $\gamma(\mu_a, \mu_b)$ vanishes faster than linearly in either of its arguments. It is also worth noticing that the scaling form (4.88) of the bistatic coefficient saturates the sum rule (2.37).

The full expression of the bistatic coefficient $\gamma(\mu_a, \mu_b)$ is obtained by taking the $\kappa \to 0$ limit of equation (4.82), using the definition (4.62). The modes $m, n = 0$ yield divergent contributions as $\kappa \to 0$, which cancel out as they should, as well as finite parts, so that we are left with the result

$$\begin{aligned}\frac{\gamma(\mu_a, \mu_b)}{\mu_a\mu_b} &= 3\tau_0 + \frac{3}{2}(\mu_a + \mu_b) + 2\sum_{n\geqslant 1}\frac{P(\sigma_n)}{N_n}[v_1(\sigma_n, x_a) + v_1(\sigma_n, x_b)]\\ &\quad - \sum_{n\geqslant 1}\frac{1}{N_n}[v_1(\sigma_n, x_a)v_1(\sigma_n, -x_b) + v_1(\sigma_n, -x_a)v_1(\sigma_n, x_b)]\\ &\quad - \frac{2}{3}\sum_{m,n\geqslant 1}\frac{\sigma_m\sigma_n}{\sigma_m + \sigma_n}\frac{P(\sigma_m)}{N_m}\frac{P(\sigma_n)}{N_n}v_1(\sigma_m, x_a)v_1(\sigma_n, x_b)\end{aligned} \tag{4.89}$$

with $\mu_a = \tanh x_a$, $\mu_b = \tanh x_b$.

The maximal value of the bistatic coefficient, which yields an absolute prediction for the diffuse reflected intensity in the normal direction, reads $\gamma(1, 1) = 4.889\,703$. This number is some 15% above the corresponding one in the case of isotropic scattering (see table 2).

4.5. Extinction lengths of azimuthal excitations

Up to this point, we have mostly investigated quantities with cylindrical symmetry around the normal to the slab, pertaining thus to the $m = 0$ sector of the azimuthal decomposition (2.8).

We now want to consider briefly the other values of the azimuthal integer m, in the regime of very anisotropic scattering. As already mentioned in section 2, all the sectors contribute, for example, to the reflected intensity, except if the incident beam is normal to the sample, or, more generally, has itself cylindrical symmetry. The situation is different in transmission through thick slabs, to which only the sector $m = 0$ contributes. The reason for this is that the intensity in the other sectors is exponentially damped inside the sample, namely

$$I^{(m)} \sim \exp(-z/L_{\text{ext}}^{(m)}) \tag{4.90}$$

where $L_{\text{ext}}^{(m)}$ is the extinction length of the azimuthal excitations in the sector m.

It is the purpose of this section to determine these lengths in the limit of very anisotropic scattering. The Legendre operator in the sector defined by the azimuthal integer m is given by equation (4.11). As a consequence, and along the lines of sections 4.3 and 4.4, we are led to study the Schrödinger equation

$$u''(\sigma, x) - (m^2 + \sigma V(x))u(\sigma, x) = 0. \tag{4.91}$$

The corresponding extinction length is given by

$$L_{\text{ext}}^{(m)} = \frac{2\ell^*}{\sigma_0^{(m)}} \tag{4.92}$$

where $\sigma_0^{(m)}$ is the smallest positive eigenvalue of equation (4.91). This is indeed the location of the first singularity of the Laplace transform of the intensity.

For large values of the azimuthal integer m, we can use the semiclassical analysis developed in appendix B. The Sommerfeld quantization formula associated with equation (4.91) reads

$$\int_{x_-}^{x_+} (-\sigma V(x) - m^2)^{1/2}\,\mathrm{d}x = \int_{\mu_-}^{\mu_+} \left(\sigma \frac{\mu}{1-\mu^2} - \frac{m^2}{(1-\mu^2)^2}\right)^{1/2} \mathrm{d}\mu \approx (n + 1/2)\pi. \tag{4.93}$$

The smallest eigenvalue $\sigma_0^{(m)}$ corresponds to setting $n = 0$ in the above formula. In first approximation we express that the argument of the square-root inside the μ-integral in (4.93) has zero as its maximal value, so that $\mu_- = \mu_+$. We thus obtain $\sigma_0^{(m)} \approx 3\sqrt{3}m^2/2$. In second approximation we expand the integrand around its maximum, which takes place for $\mu \approx \sqrt{3}/3$. We obtain after some algebra the following next-to-leading order estimate

$$\sigma_0^{(m)} \approx (3\sqrt{3}/2)(m^2 + m\sqrt{2}) \tag{4.94}$$

hence

$$L_{\text{ext}}^{(m)} \approx \frac{4\ell^*}{3\sqrt{3}(m^2 + m\sqrt{2})} \qquad (m \gg 1). \tag{4.95}$$

This semiclassical estimate gives accurate numbers of the whole spectrum of extinction lengths, down to the largest one, $L_{\text{ext}}^{(1)}$. Indeed equation (4.95) yields $L_{\text{ext}}^{(1)} \approx 0.318\,861\ell^*$, whereas the exact numerical value reads $L_{\text{ext}}^{(1)} = 0.282\,916\,9\ell^*$.

For a large but finite anisotropy, the lengths $L_{\text{ext}}^{(m)}$ follow the universal law (4.95) only for $m < m^* \sim \sqrt{\tau^*} \sim 1/\Theta_{\text{rms}}$. For larger values of the azimuthal number, the $L_{\text{ext}}^{(m)}$ become non-universal numbers of order ℓ. This crossover is expected on physical grounds. Indeed large azimuthal numbers $m \gg m^*$ correspond to an angular resolution $\delta\Theta \ll \Theta_{\text{rms}}$, so that the details of the cross section matter in this regime.

5. Discussion

In this paper we have considered several aspects of multiple anisotropic scattering of scalar waves. We have considered the geometry of an optically thick slab, of thickness $L \gg \ell^*$. Our main goal has been to investigate in a quantitative way the effects of the anisotropy of the scattering cross section, and of the internal reflections at the boundaries of the sample, due to an optical index mismatch.

The general results derived in section 2 show that, in first approximation, quantities only depend on the anisotropy through the transport mean free path ℓ^*. This is especially the case for the angle-resolved transmission through a thick slab (2.47), for the thickness of a skin layer, $z_0 = \tau_0 \ell^*$, and for the width (2.68) of the enhanced backscattering cone. The present work thus confirms on a firm basis that the scaling behaviour of these quantities is qualitatively explained within the diffusion approximation, which amounts to only considering the long-distance diffusive character of the propagation of radiation in a turbid medium. The scaling law in $1/\ell^*$ of the width of the cone was derived long ago within the diffusion approximation [31, 32]. The results of section 2.7 concerning the dependence of the extinction length with respect to Q and a can be directly compared with the prediction of the diffusion approximation [12]. Within this framework, all extinction effects, provided they are small enough, can be coded in a single parameter, namely the *mass* $\mathcal{M}$ such that

$$\mathcal{M}^2 = q^2 + \mathrm{i}\Omega + \frac{1}{L_{\mathrm{abs}}^2} \tag{5.1}$$

where q is the transverse wavevector, L_{abs} is the absorption length, and $\Omega = (\omega - \omega')/D_{\mathrm{phys}}$ represents the properly dimensioned contribution of a small frequency shift between the advanced and the retarded amplitude propagators which build up the *diffuson*. The inverse extinction length is then equal to the real part of the complex mass $\mathcal{M}$. Our results fully agree with equation (5.1), with $\mathcal{M} \approx s_0/\ell$, $q = Q/\ell$, and L_{abs} as in equation (2.71).

We then investigated in detail to what extent observable quantities are universally described by their explicit dependence on ℓ^* recalled above, and to what extent they still depend on details of the scattering cross section mechanism. As recalled in the introduction, this question is beyond the scope of the diffusion approximation, and requires a careful treatment using the radiative transfer theory, at least in the regime $\ell \gg \lambda_0$. For the diffuse reflected or transmitted intensity, and for the width of the enhanced backscattering cone, the detailed structure of the scattering mechanism only contributes a small effect, entirely contained in prefactors of the laws mentioned above, such as the constant τ_0, or the functions $\tau_1(\mu)$ and $\gamma(\mu, \mu')$.

Two regimes of interest allow for more quantitative results.

(i) The regime of a large index mismatch, where the boundaries of the sample almost act as perfect mirrors, is considered in section 3. Our results (3.9), (3.12) are identical to those derived in [15], in the case of isotropic scattering. Therefore the quantities we have considered do not depend *at all* on the scattering cross section in this regime. This can be understood as follows. Since the thickness $z_0 \approx 4\ell^*/(3\mathcal{T})$ of a skin layer is very large, the radiation undergoes many scattering events near the boundaries before it leaves the medium, so that the details of every single scattering event are washed out.

(ii) In the absence of internal reflections, we have considered in detail the regime of very anisotropic scattering. In section 4 we have presented an exact analytical treatment of the radiative transfer problem in this regime. We have obtained the results (4.50), (4.57), (4.89) which determine the diffuse reflected and transmitted light for a thick slab. These results are given in terms of the eigenvalues and eigenfunctions of the one-dimensional Schrödinger equations, which are accessible both numerically, via the partial-wave expansion of appendix A, and analytically in the limit of large quantum numbers, via the semiclassical analysis of appendix B. It is *a priori* possible to extend this exact treatment to the scaling behaviour of the shape of the enhanced backscattering cone. The general structure of the equations to be solved shows that we have $\gamma_C(Q) \approx F(Q\tau^*)$, in a whole scaling region defined by $Q \ll 1$ and $\tau^* \gg 1$. By inserting the numerical values of table 2 into the expansion (2.66), we get at once $F(0) = \gamma(1,1) = 4.889\,703$ and $F'(0) = -\tau_1(1)^2/3 = -8.801\,66$. The exact determination of the full scaling function F would amount to solving a self-consistent inhomogeneous equation of the type (4.61), albeit with the full Legendre operator instead of one-dimensional second-order derivative. The wings of the cone, starting around values of Q of order unity, will depend on the details of the scattering cross section, even in the regime of very anisotropic scattering.

Our exact treatment of the radiative transfer problem in the very anisotropic regime, based on the expansion (4.8), is expected to be valid in the regime $\Theta_{\mathrm{rms}} \ll 1$ of a broad *universality class* of phase functions. Although this universality class cannot be easily characterized, we can assert that it contains at least the phase functions scaling as

$$p(\Theta) \approx \Phi(\Theta/\Theta_{\mathrm{rms}}) \tag{5.2}$$

such that the scaling function Φ has a finite second moment. This restrictive definition does not encompass *a priori* the Lorentzian-squared phase function (4.4), which has a logarithmically divergent second moment, as already mentioned in section 4.1. The same remark holds for the so-called Henyey–Greenstein phase function

$$p(\Theta) = \frac{1-g^2}{(1-2g\cos\Theta+g^2)^{3/2}} \tag{5.3}$$

often used in numerical investigations [3, 17], for which the second-moment integral is linearly divergent.

The discussion of the dependence of quantities on the details of the scattering mechanism is summarized in table 2, where we compare the numerical values of the dimensionless absolute prefactors of five characteristic quantities, for isotropic scattering and for very anisotropic scattering. The relative differences, shown in the last row, are very small in most cases. Some other quantities, such as the shape of the enhanced backscattering cone, or the spectrum of extinction lengths of the azimuthal excitations, exhibit universal behaviour in the very anisotropic regime only in a limited range, corresponding to a low enough angular resolution ($\delta\Theta \gg \Theta_{\mathrm{rms}}$), so that the details of the scattering cross section do not matter.

Finally, we can compare our universal results in the very anisotropic scattering regime, for some of the quantities listed in table 2, with the outcomes of numerical approaches. Van de Hulst [3, 17] has investigated in a systematic way the dependence of various quantities on anisotropy, for several commonly used phenomenological forms of the phase function, including especially the Henyey–Greenstein phase function (5.3). The data on the skin-layer thickness reported in [3] show that, as a function of anisotropy, τ_0 varies from 0.7104 (isotropic scattering) to 0.7150 (moderate anisotropy), passing a minimum of 0.7092 (weak anisotropy). The trend shown by these data suggests that our universal value 0.718 211 is actually an absolute upper bound for τ_0. Numerical data concerning $\tau_1(1)$ is also available. Van de Hulst [17] has extrapolated two series of data, concerning the Henyey–Greenstein phase function (5.3), which admit a common limit for very anisotropic scattering ($g \to 1$).

According to the analysis of section 4, this limit reads in our language $\tau_1(1)/4 = 1.284\,645$, whereas [17] gives the two slightly different estimates 1.273 ± 0.002 and 1.274 ± 0.007. The agreement is satisfactory, although it cannot be entirely excluded that the observed 0.8% relative difference can be a small but genuine non-universality effect. Indeed, as mentioned above, the Henyey–Greenstein phase function (5.3) might not belong to the universality class where our approach holds true. The same remark applies to a less complete set of data [17] concerning the intensity $\gamma(1, 1)$ of reflected light at normal incidence.

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Appendix A. Partial-wave expansions

In this appendix we describe a numerical algorithm based on a partial-wave expansion, that we have used to determine the eigenvalues and eigenfunctions of the Schrödinger equations involved in section 4.

We first consider the Schrödinger equation (4.23). Going back to the μ-variable, this equation reads

$$(\Delta^{(0)} - \sigma\mu)v_1(\sigma, \mu) = 0 \tag{A.1}$$

where $\Delta^{(0)}$ is the Legendre operator in the φ-independent sector, defined in equation (4.11). It is natural to expand the function $v_1(\sigma, \mu)$ in the Legendre polynomials

$$v_1(\sigma, \mu) = \sum_{\ell \geqslant 0} a_\ell(\sigma) P_\ell(\mu). \tag{A.2}$$

Indeed these polynomials are eigenfunctions of $\Delta^{(0)}$, namely

$$\Delta^{(0)} P_\ell = -\ell(\ell + 1) P_\ell \tag{A.3}$$

and the product $\mu P_\ell(\mu)$ has the following expression

$$(2\ell + 1)\mu P_\ell(\mu) = (\ell + 1) P_{\ell+1}(\mu) + \ell P_{\ell-1}(\mu) \tag{A.4}$$

so that (A.1) amounts to the following three-term recursion relation

$$\ell(\ell + 1)a_\ell + \sigma\left(\frac{\ell + 1}{2\ell + 3} a_{\ell+1} + \frac{\ell}{2\ell - 1} a_{\ell-1}\right) = 0. \tag{A.5}$$

When σ is one of the eigenvalues σ_n, (A.5) has an acceptable solution $\{a_\ell(\sigma)\}$, decaying to zero for large ℓ. The quantities needed in section 4 can then be evaluated as follows. The normalization condition (4.25) becomes

$$\sum_{\ell \geqslant 0} a_\ell(\sigma_n) = 1 \tag{A.6}$$

since $P_\ell(1) = 1$. The squared norms N_n of the eigenfunctions read

$$N_n = 4 \sum_{\ell \geqslant 0} \frac{\ell + 1}{(2\ell + 1)(2\ell + 3)} a_\ell(\sigma_n) a_{\ell+1}(\sigma_n) \tag{A.7}$$

as a consequence of the normalization of the Legendre polynomials

$$\int_{-1}^{1} \frac{\mathrm{d}\mu}{2} P_k(\mu) P_\ell(\mu) = \frac{\delta_{k,\ell}}{2\ell + 1}. \tag{A.8}$$

Finally, the non-trivial mixed Wronskians $F(\pm\sigma_n)$ read

$$F(-\sigma_n) = \frac{1}{F(\sigma_n)} = v_1(\sigma_n, \mu = -1) = \sum_{\ell \geqslant 0} (-1)^\ell a_\ell(\sigma_n) \tag{A.9}$$

since $P_\ell(-1) = (-1)^\ell$.

We now consider the κ-dependent Schrödinger equation (4.64), namely

$$(\Delta^{(0)} - \sigma\mu + \kappa^2(\mu^2 - 1)) v_1(\kappa, \sigma, \mu) = 0. \tag{A.10}$$

We again expand the wavefunction over the Legendre polynomials

$$v_1(\kappa, \sigma, \mu) = \sum_{\ell \geqslant 0} a_\ell(\kappa, \sigma) P_\ell(\mu). \tag{A.11}$$

By iterating (A.4) twice, we obtain the following five-term recursion

$$\begin{aligned} \ell(\ell+1)a_\ell + \sigma \left(\frac{\ell+1}{2\ell+3} a_{\ell+1} + \frac{\ell}{2\ell-1} a_{\ell-1} \right) - \kappa^2 \biggl(\frac{\ell(\ell-1)}{(2\ell-1)(2\ell-3)} a_{\ell-2} \\ + \frac{2(1-\ell-\ell^2)}{(2\ell-1)(2\ell+3)} a_\ell + \frac{(\ell+1)(\ell+2)}{(2\ell+3)(2\ell+5)} a_{\ell+2} \biggr) = 0. \end{aligned} \tag{A.12}$$

Equations (A.6), (A.7), (A.9) still hold true.

We finally consider the wave equation

$$(\Delta - \sigma\mu) v_1(\sigma, \mu, \varphi) = 0 \tag{A.13}$$

where Δ is the full Legendre operator, defined in (4.9). Since the potential does not involve the azimuthal angle φ explicitly, we look for a solution v_1 proportional to $\mathrm{e}^{\mathrm{i}m\varphi}$, with $m \geqslant 0$ being an integer. It is now natural to expand the function $v_1(\sigma, \mu, \varphi)$ in the Legendre functions $P_{\ell,m}(\mu)$, namely

$$v_1(\sigma, \mu, \varphi) = \mathrm{e}^{\mathrm{i}m\varphi} \sum_{\ell \geqslant m} a_{\ell,m}(\sigma) P_{\ell,m}(\mu). \tag{A.14}$$

These functions obey

$$\Delta(P_{\ell,m}(\mu)\mathrm{e}^{\mathrm{i}m\varphi}) = -\ell(\ell+1) P_{\ell,m}(\mu)\mathrm{e}^{\mathrm{i}m\varphi} \tag{A.15}$$

and the product $\mu P_{\ell,m}(\mu)$ has the following expression

$$(2\ell+1)\mu P_{\ell,m}(\mu) = (\ell+1-m) P_{\ell+1,m}(\mu) + (\ell+m) P_{\ell-1,m}(\mu) \tag{A.16}$$

so that (A.13) amounts to the three-term recursion relation

$$\ell(\ell+1)a_{\ell,m} + \sigma \left(\frac{\ell+m+1}{2\ell+3} a_{\ell+1,m} + \frac{\ell-m}{2\ell-1} a_{\ell-1,m} \right) = 0. \tag{A.17}$$

The recursion equations (A.5), (A.12), (A.17) are easily implemented numerically.

Appendix B. Semiclassical analysis

The outcomes of the semiclassical analysis presented in this appendix are used at various places in section 4. We first consider the Schrödinger equation (4.23). For large values of the complex parameter σ, we look for rapidly varying solutions of the form

$$u \approx \Phi(x)^{-1/2} \exp \int^{x} \Phi(y)\, \mathrm{d}y \tag{B.1}$$

with

$$\Phi(x)^2 = \sigma V(x). \tag{B.2}$$

This approach is analogous to the WKB approximation in quantum mechanics, since the condition $\sigma \gg 1$ is equivalent to $\hbar$ being small.

Because the potential $V(x)$ is odd, we can restrict the analysis to the domain $\mathrm{Re}\,\sigma > 0$. We introduce the notation

$$K = \sqrt{\sigma}. \tag{B.3}$$

Equation (B.2) has real solutions for $x > 0$. We set

$$p(x) = \sqrt{V(x)} \qquad (x > 0) \tag{B.4}$$

so that $\Phi(x) = Kp(x)$. Equation (B.1) thus yields the basis of functions

$$u_{\pm}(x) \approx \frac{1}{(Kp(x))^{1/2}} \exp(\pm K I(x)) \tag{B.5}$$

with

$$I(x) = \int_{x}^{\infty} p(y)\, \mathrm{d}y \qquad (x > 0). \tag{B.6}$$

The above functions $u_{\pm}(x)$ are exponentially blowing up or decaying. The domains $x > 0$ and $\sigma > 0$ (and similarly $x < 0$ and $\sigma < 0$) are said to be *classically forbidden*.

On the other hand, for $x < 0$, equation (B.2) has imaginary solutions. We set

$$q(x) = \sqrt{-V(x)} \qquad (x < 0) \tag{B.7}$$

so that $\Phi(x) = \mathrm{i}Kq(x)$. Equation (B.1) thus yields the basis of functions

$$u_{\pm}(x) \approx \frac{1}{(Kq(x))^{1/2}} \exp(\pm \mathrm{i} K I(x)) \tag{B.8}$$

with

$$I(x) = \int_{-\infty}^{-x} q(y)\, \mathrm{d}y \qquad (x < 0). \tag{B.9}$$

The above functions $u_{\pm}(x)$ are oscillating and bounded, up to the prefactor in $q(x)^{-1/2}$. The domains $x < 0$ and $\sigma > 0$ (and similarly $x > 0$ and $\sigma < 0$) are said to be *classically allowed*.

The difficulty of the semiclassical analysis comes from the existence of three *turning points*, namely $x = 0$ and $x \to \pm\infty$, where the momentum variable $p(x)$ or $q(x)$ vanishes. The estimates (B.5), (B.8) lose their meaning in the vicinity of the turning points, where a more careful analysis is required, to be presented now.

We first investigate the basis of functions $\{u_1, u_2\}$ for $x < 0$. For $x \to -\infty$ and $K \to \infty$, the Schrödinger equation (4.23) assumes the simpler form

$$u'' + (2K\mathrm{e}^{x})^2 u = 0. \tag{B.10}$$

A basis of solutions to this equation is given by the Bessel functions $J_0(z)$ and $N_0(z)$, with $z = 2K\mathrm{e}^x$ being a scaling variable. The boundary conditions (4.24) have to match the known small-z behaviour of the Bessel functions, hence

$$\begin{aligned} u_1(x) &\approx J_0(2K\mathrm{e}^x) \\ u_2(x) &\approx (\pi/2)N_0(2K\mathrm{e}^x) - (\ln K + \gamma_E)J_0(2K\mathrm{e}^x) \qquad (x \to -\infty) \end{aligned} \tag{B.11}$$

where γ_E denotes Euler's constant. The known large-z behaviour of the Bessel functions fixes the amplitudes of the integrals in (B.8) for u_1 and u_2, namely

$$\begin{aligned} u_1(x) &\approx \left(\frac{2}{\pi K q(x)}\right)^{1/2} \cos(KI(x) - \pi/4) \\ u_2(x) &\approx \left(\frac{2}{\pi K q(x)}\right)^{1/2} [(\pi/2)\sin(KI(x) - \pi/4) - (\ln K + \gamma_E)\cos(KI(x) - \pi/4)]. \end{aligned} \tag{B.12}$$

On the other hand, for $x \to 0$ and $K \to \infty$, the Schrödinger equation (4.23) assumes the simpler form

$$u'' + K^2 x u = 0 \tag{B.13}$$

which is equivalent to Airy's equation. A basis of solutions is given by the Airy functions $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$, with $z = K^{2/3}x$ being again a scaling variable. The known behaviour for $z \to -\infty$ of the Airy functions has to match equation (B.12), hence

$$\begin{aligned} u_1(x) &\approx 2^{1/2}K^{-1/3}[\sin(KI)\,\mathrm{Ai}(K^{2/3}x) + \cos(KI)\,\mathrm{Bi}(K^{2/3}x)] \\ u_2(x) &\approx 2^{1/2}K^{-1/3}\{(\pi/2)[-\cos(KI)\,\mathrm{Ai}(K^{2/3}x) + \sin(KI)\,\mathrm{Bi}(K^{2/3}x)] \\ &\quad - (\ln K + \gamma_E)[\sin(KI)\,\mathrm{Ai}(K^{2/3}x) + \cos(KI)\,\mathrm{Bi}(K^{2/3}x)]\} \end{aligned} \tag{B.14}$$

for $x \to 0$, with

$$I = I(0) = \int_0^\infty p(x)\,\mathrm{d}x = \int_0^1 \mathrm{d}\mu \left(\frac{\mu}{1-\mu^2}\right)^{1/2} = \frac{\sqrt{\pi}}{2}\frac{\Gamma(3/4)}{\Gamma(5/4)} = 1.198\,140 \tag{B.15}$$

this definition being consistent with equations (B.6), (B.9).

We now investigate in a similar way the functions $\{v_1, v_2\}$ for $x > 0$. For $x \to \infty$ and $K \to \infty$, a basis of solutions is given by the modified Bessel functions $I_0(z)$ and $K_0(z)$, with $z = 2K\mathrm{e}^{-x}$. The boundary conditions (4.25) have to match the known small-z behaviour of the Bessel functions, hence

$$\begin{aligned} v_1 &\approx I_0(2K\mathrm{e}^{-x}) \\ v_2 &\approx K_0(2K\mathrm{e}^{-x}) + (\ln K + \gamma_E)I_0(2K\mathrm{e}^{-x}) \qquad (x \to \infty). \end{aligned} \tag{B.16}$$

The known large-z behaviour of the Bessel functions only fixes the amplitude of the solution u_+ of (B.5) for v_1 and v_2, namely

$$\begin{aligned} v_1(x) &\approx \left(\frac{1}{2\pi K p(x)}\right)^{1/2} \mathrm{e}^{KI(x)} \\ v_2(x) &\approx (\ln K + \gamma_E)v_1(x) \qquad (x > 0). \end{aligned} \tag{B.17}$$

Finally, the known behaviour of the Airy functions as $z \to \infty$ yields

$$\begin{aligned} v_1(x) &\approx 2^{1/2}K^{-1/3}\,\mathrm{Ai}(K^{2/3}x)\mathrm{e}^{KI} \\ v_2(x) &\approx (\ln K + \gamma_E)v_1(x) \qquad (x \to 0). \end{aligned} \tag{B.18}$$

The above expressions (B.14), (B.18) of both bases of solutions as $x \to 0$ and $K \to \infty$ allow us to derive the following semiclassical estimates for the elements of the transfer matrix introduced in equation (4.29)

$$\begin{aligned}
&F(\sigma) \approx [\sin(KI) - (2/\pi)(\ln K + \gamma_E)\cos(KI)]\mathrm{e}^{KI} \\
&G(\sigma) \approx -(2/\pi)\cos(KI)\mathrm{e}^{KI} \\
&H(\sigma) \approx (\ln K + \gamma_E)F(\sigma) \\
&F(-\sigma) \approx (\ln K + \gamma_E)G(\sigma) \qquad (\sigma = K^2 \to \infty).
\end{aligned} \tag{B.19}$$

The estimate for $G(\sigma)$ directly yields the following expressions for its factors $P(\pm\sigma)$, defined in equation (4.41):

$$\begin{aligned}
&P(\sigma) \approx (3/\pi)^{1/2}\frac{\mathrm{e}^{KI}}{K^2} \\
&P(-\sigma) \approx -2(3/\pi)^{1/2}\frac{\cos(KI)}{K^2} \qquad (\sigma = K^2 \to \infty).
\end{aligned} \tag{B.20}$$

We also obtain from (B.19) an estimate of the eigenvalues $\sigma_n = K_n^2$, which are the zeros of $G(\sigma)$, in the form

$$K_n \approx (n + 1/2)\frac{\pi}{I} \qquad (n \gg 1). \tag{B.21}$$

This semiclassical formula gives a very accurate description of the whole spectrum of the Schrödinger equation (4.23). Indeed the relative error is maximal for the first non-zero eigenvalue, for which (B.21) predicts $K_1 \approx 3.933\,086$, i.e. some 3.2% above the exact numerical value $K_1 = 3.811\,562$.

The semiclassical expression of the squared norms N_n can be evaluated by inserting the estimates (B.19) into the identity (C.6). We thus obtain

$$N_n \approx -\frac{I}{\pi K_n}\mathrm{e}^{2K_n I} \qquad (n \gg 1). \tag{B.22}$$

In section 4 we also need the expression of the Mellin transform of the Airy function $\mathrm{Ai}(x)$, namely

$$m(s) = \int_0^\infty x^s\,\mathrm{Ai}(x)\,\mathrm{d}x \qquad (\mathrm{Re}\, s > -1) \tag{B.23}$$

which we have not found in standard handbooks. The Airy equation implies the functional equation

$$m(s+3) = (s+1)(s+2)m(s) \tag{B.24}$$

whose correctly normalized solution is

$$m(s) = 3^{-s/3}\frac{\Gamma(s)}{\Gamma(s/3)}. \tag{B.25}$$

We now consider the deformed Schrödinger equation (4.63), where both σ and κ are non-zero. It turns out that only the spectrum of that wave equation will be needed. Hence we can content ourselves with the Sommerfeld quantization formula. A similar treatment is used in section 4.5 for the full Legendre operator.

The Sommerfeld formula reads

$$\int_{x_-}^{x_+}(-\sigma V(x) - \kappa^2 W(x))^{1/2}\,\mathrm{d}x = \int_{\mu_-}^{\mu_+}\left(\sigma\frac{\mu}{1-\mu^2} - \kappa^2\right)^{1/2}\mathrm{d}\mu \approx (n+1/2)\pi \tag{B.26}$$

the integral being extended over the classically allowed domain, where the square root is real.

The implicit equation (B.26) for the semiclassical estimate of the eigenvalues $\pm\sigma_n(\kappa)$ can be investigated in several limiting cases of interest. For small values of κ, the integrand can be expanded in a straightforward way. We thus obtain

$$\sigma_n(\kappa)^{1/2} \approx K_n(\kappa) = (n+1/2)\,\frac{\pi}{I}\left(1+\frac{2\kappa^2}{3\pi(n+1/2)^2}+\cdots\right) \qquad (n\gg 1,\ \kappa \ll 1). \tag{B.27}$$

This expression confirms the general result (4.65). On the other hand, for $\sigma = 0$, it can be deduced from equation (B.26) that the spheroidal equation (4.70) has imaginary eigenvalues of the form $\kappa = \pm\mathrm{i}\xi_n$, asymptotically given by the semiclassical estimate

$$\xi_n \approx (n+1/2)\pi/2 \qquad (n \gg 1). \tag{B.28}$$

Appendix C. Useful identities on the mixed Wronskians

In this appendix, we derive the identity (C.6) used in section 4, and more generally we give alternative expressions for the derivatives with respect to the spectral variable σ of the mixed Wronskians $F(\sigma)$, $G(\sigma)$ and $H(\sigma)$, which enter the transfer matrix (4.29).

To do so, we start by considering the derivatives

$$U_\alpha(\sigma,x) = \frac{\partial u_\alpha(\sigma,x)}{\partial\sigma} \qquad (\alpha = 1,2) \tag{C.1}$$

which obey the inhomogeneous Schrödinger equation

$$U_\alpha''(\sigma,x) - \sigma V(x)U_\alpha(\sigma,x) = V(x)u_\alpha(\sigma,x) \qquad (\alpha=1,2). \tag{C.2}$$

These equations can be solved explicitly by *varying the constants*, along the lines of sections 4.3 and 4.4. We thus obtain

$$\begin{aligned} U_1(\sigma,x) &= -u_1(\sigma,x)\int_{-\infty}^{x} u_1(\sigma,y)u_2(\sigma,y)V(y)\,\mathrm{d}y + u_2(\sigma,x)\int_{-\infty}^{x} u_1^2(\sigma,y)V(y)\,\mathrm{d}y \\ U_2(\sigma,x) &= -u_1(\sigma,x)\int_{-\infty}^{x} u_2^2(\sigma,y)V(y)\,\mathrm{d}y + u_2(\sigma,x)\int_{-\infty}^{x} u_1(\sigma,y)u_2(\sigma,y)V(y)\,\mathrm{d}y. \end{aligned} \tag{C.3}$$

By taking the $x \to \infty$ limit of the above expressions, and using the asymptotic behaviour (4.30), we get the following expressions

$$\begin{aligned} \frac{\mathrm{d}F(\sigma)}{\mathrm{d}\sigma} &= G(\sigma)N_{22}(\sigma) + F(\sigma)N_{12}(\sigma) \\ \frac{\mathrm{d}G(\sigma)}{\mathrm{d}\sigma} &= -F(\sigma)N_{11}(\sigma) - G(\sigma)N_{12}(\sigma) \\ \frac{\mathrm{d}H(\sigma)}{\mathrm{d}\sigma} &= -F(\sigma)N_{11}(\sigma) - G(\sigma)N_{12}(\sigma) \\ \frac{\mathrm{d}F(-\sigma)}{\mathrm{d}\sigma} &= -F(-\sigma)N_{12}(\sigma) - H(\sigma)N_{11}(\sigma) \end{aligned} \tag{C.4}$$

with the definition

$$N_{\alpha\beta}(\sigma) = \int_{-\infty}^{\infty} u_\alpha(\sigma,x)u_\beta(\sigma,x)V(x)\,\mathrm{d}x \qquad (\alpha,\beta=1,2). \tag{C.5}$$

On the spectrum, i.e. for $\sigma = \sigma_n$, we have $G(\sigma_n) = 0$, by definition. Furthermore, equation (4.34) implies $N_{11}(\sigma_n) = N_n / F^2(\sigma_n)$, hence the identity

$$\left(\frac{\mathrm{d}G(\sigma)}{\mathrm{d}\sigma}\right)_{\sigma=\sigma_n} = -\frac{N_n}{F(\sigma_n)} = -N_n F(-\sigma_n) \qquad \text{(C.6)}$$

with N_n being the squared norm of the eigenfunction $v_1(\sigma_n, x)$, defined in equation (4.35). The identity (C.6) is very general. It also holds for the κ-dependent Schrödinger equation (4.63).

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